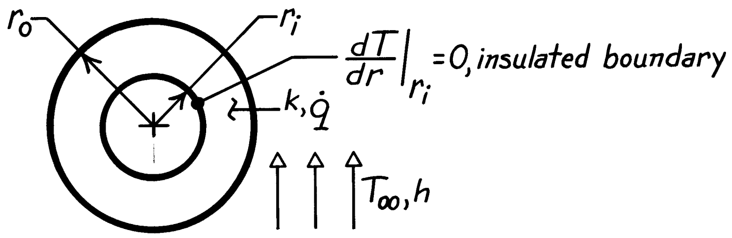
PROBLEM 3.96

KNOWN: Cylindrical shell with uniform volumetric generation is insulated at inner surface and exposed to convection on the outer surface.

FIND: (a) Temperature distribution in the shell in terms of r_i, r_o, $\dot{q}$, h, T_∞ and k, (b) Expression for the heat rate per unit length at the outer radius, $q'(r_o)$.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) One-dimensional radial (cylindrical) conduction in shell, (3) Uniform generation, (4) Constant properties.

ANALYSIS: (a) The general form of the temperature distribution and boundary conditions are

$$T(r) = -\frac{\dot{q}}{4k} r^2 + C_1 \ln r + C_2$$

at $r = r_i$: $\qquad \left.\frac{dT}{dr}\right)_{r_i} = 0 = -\frac{\dot{q}}{2k} r_i + C_1 \frac{1}{r_i} + 0 \qquad C_1 = \frac{\dot{q}}{2k} r_i^2$

at $r = r_o$: $\qquad -k \left.\frac{dT}{dr}\right)_{r_o} = h\left[T(r_o) - T_\infty\right] \qquad$ surface energy balance

$$-k\left[-\frac{\dot{q}}{2k} r_o + \left(\frac{\dot{q}}{2k} r_i^2 \cdot \frac{1}{r_o}\right)\right] = h\left[-\frac{\dot{q}}{4k} r_o^2 + \left(\frac{\dot{q}}{2k} r_i^2\right)\ln r_o + C_2 - T_\infty\right]$$

$$C_2 = -\frac{\dot{q} r_o}{2h}\left[1 - \left(\frac{r_i}{r_o}\right)^2\right] + \frac{\dot{q} r_o^2}{2k}\left[\frac{1}{2} - \left(\frac{r_i}{r_o}\right)^2 \ln r_o\right] + T_\infty$$

Hence,

$$T(r) = \frac{\dot{q}}{4k}\left(r_o^2 - r^2\right) + \frac{\dot{q} r_i^2}{2k} \ln\left(\frac{r}{r_o}\right) - \frac{\dot{q} r_o}{2h}\left[1 - \left(\frac{r_i}{r_o}\right)^2\right] + T_\infty . \qquad <$$

(b) From an overall energy balance on the shell,

$$q_r'(r_o) = \dot{E}_g' = \dot{q}\pi\left(r_o^2 - r_i^2\right). \qquad <$$

Alternatively, the heat rate may be found using Fourier's law and the temperature distribution,

$$q_r'(r) = -k(2\pi\, r_o)\left.\frac{dT}{dr}\right)_{r_o} = -2\pi\, k r_o\left[-\frac{\dot{q}}{2k} r_o + \frac{\dot{q} r_i^2}{2k}\,\frac{1}{r_o} + 0 + 0\right] = \dot{q}\pi\left(r_o^2 - r_i^2\right)$$