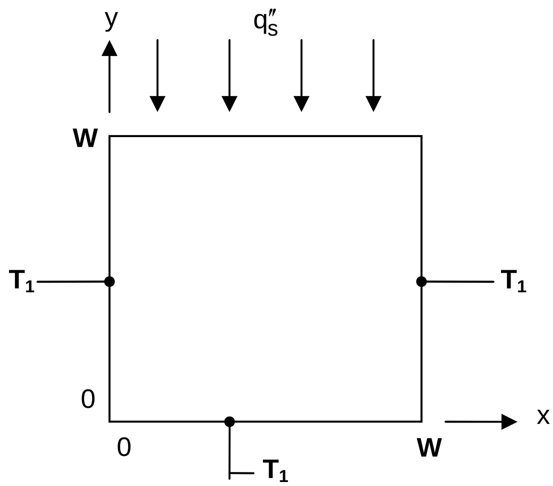
PROBLEM 4.8

KNOWN: Boundary conditions on four sides of a square plate.

FIND: Expressions for shape factors associated with the *maximum* and *average* top surface temperatures. Values of these shape factors. The maximum and average temperatures for specified conditions.

SCHEMATIC:



ASSUMPTIONS: (1) Two-dimensional, steady-state conduction, (2) Constant properties.

ANALYSIS: We must first find the temperature distribution as in Problem 4.5. Problem 4.5 differs from the problem solved in Section 4.2 only in the boundary condition at the top surface. Defining $\theta = T - T_\infty$, the differential equation and boundary conditions are

$$\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} = 0$$

$$\theta(0,y) = 0 \quad \theta(L,y) = 0 \quad \theta(x,0) = 0 \quad k\left.\frac{\partial\theta}{\partial y}\right|_{y=W} = q_s'' \qquad (1a,b,c,d)$$

The solution is identical to that in Section 4.2 through Equation (4.11),

$$\theta = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{L}\sinh\frac{n\pi y}{L} \qquad (2)$$

To determine C_n, we now apply the top surface boundary condition, Equation (1d). Differentiating Equation (2) yields

$$\left.\frac{\partial\theta}{\partial y}\right|_{y=W} = \sum_{n=1}^{\infty} C_n \frac{n\pi}{L}\sin\frac{n\pi x}{L}\cosh\frac{n\pi W}{L} \qquad (3)$$

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Substituting this into Equation (1d) results in

$$\frac{q_s''}{k} = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L} \qquad (4)$$

where $A_n = C_n(n\pi/L)\cosh(n\pi W/L)$. The principles expressed in Equations (4.13) through (4.16) still apply, but now with reference to Equation (4) and Equation (4.14), we should choose $f(x) = q_s''/k$, $g_n(x) = \sin\frac{n\pi x}{L}$. Equation (4.16) then becomes

$$A_n = \frac{\frac{q_s''}{k}\int_0^L \sin\frac{n\pi x}{L}dx}{\int_0^L \sin^2\frac{n\pi x}{L}dx} = \frac{q_s''}{k}\frac{2}{\pi}\frac{(-1)^{n+1}+1}{n}$$

Thus

$$C_n = 2\frac{q_s''L}{k}\frac{(-1)^{n+1}+1}{n^2\pi^2\cosh(n\pi W/L)} \qquad (5)$$

The solution is given by Equation (2) with C_n defined by Equation (5). We now proceed to evaluate the shape factors.

(a) The maximum top surface temperature occurs at the midpoint of that surface, x = W/2, y = W. From Equation (2) with L = W,

$$\theta(W/2,W) = T_{2,max} - T_1 = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi}{2}\sinh n\pi = \sum_{n\ odd} C_n(-1)^{(n-1)/2}\sinh n\pi$$

where

$$C_n = 2\frac{q_s''W}{k}\frac{(-1)^{n+1}+1}{n^2\pi^2\cosh n\pi}$$

Thus

$$S_{max} = \frac{q_s''Wd}{k(T_{2,max} - T_1)} = \left[\frac{2}{d}\sum_{n\ odd}\frac{(-1)^{n+1}+1}{n^2\pi^2}(-1)^{(n-1)/2}\tanh n\pi\right]^{-1} = \left[\frac{4}{d}\sum_{n\ odd}\frac{(-1)^{(n-1)/2}}{n^2\pi^2}\tanh n\pi\right]^{-1} \quad <$$

where d is the depth of the rectangle into the page.

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(b) The average top surface temperature is given by

$$\bar{\theta}(y = W) = \bar{T}_2 - T_1 = \sum_{n=1}^{\infty} C_n \frac{1}{W} \int_0^W \sin \frac{n\pi x}{W}\, dx \sinh n\pi = \sum_{n=1}^{\infty} C_n \frac{1 - (-1)^n}{n\pi} \sinh n\pi$$

Thus

$$\bar{S} = \frac{q_s''Wd}{k(\bar{T}_2 - T_1)} = \left[\frac{2}{d}\sum_{n=1}^{\infty} \frac{[(-1)^{n+1} + 1][1 - (-1)^n]}{n^3\pi^3} \tanh n\pi\right]^{-1} = \left[\frac{8}{d}\sum_{n\ \text{odd}} \frac{1}{n^3\pi^3} \tanh n\pi\right]^{-1} \qquad <$$

(c) Evaluating the expressions for the shape factors yields

$$\frac{S_{max}}{d} = \left[4 \sum_{n\ \text{odd}} \frac{(-1)^{(n-1)/2}}{n^2\pi^2} \tanh n\pi\right]^{-1} = 2.70 \qquad <$$

$$\frac{\bar{S}}{d} = \left[8 \sum_{n\ \text{odd}} \frac{1}{n^3\pi^3} \tanh n\pi\right]^{-1} = 3.70 \qquad <$$

The temperatures can then be found from

$$T_{2,max} = T_1 + \frac{q}{S_{max} k} = T_1 + \frac{q_s''Wd}{S_{max} k} = 0°\text{C} + \frac{1000\ \text{W/m}^2 \times 0.01\ \text{m}}{2.70 \times 20\ \text{W/m} \cdot \text{K}} = 0.19°\text{C} \qquad <$$

$$\bar{T}_2 = T_1 + \frac{q}{\bar{S}k} = T_1 + \frac{q_s''Wd}{\bar{S}k} = 0°\text{C} + \frac{1000\ \text{W/m}^2 \times 0.01\ \text{m}}{3.70 \times 20\ \text{W/m} \cdot \text{K}} = 0.14°\text{C} \qquad <$$