PROBLEM 4.56

KNOWN: Dimensions and thermal conductivity distribution within a two-dimensional solid. Applied boundary conditions.

FIND: (a) Spatially-averaged thermal conductivity and heat rate per unit length based upon this value, (b) Heat rate per unit length for case 1 boundary conditions and comparison to estimated heat rate per unit length based upon the spatially-averaged thermal conductivity, (c) Heat rate per unit length for case 2 boundary conditions and comparison to estimated heat rate per unit length based upon the spatially-averaged thermal conductivity.

ASSUMPTIONS: Steady-state, one-dimensional heat transfer.

ANALYSIS: (a) The thermal conductivity varies only in the *x*-direction. Hence,

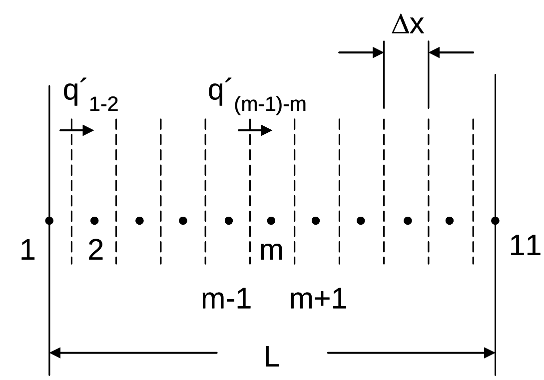
$$\bar{k} = \frac{1}{L}\int_{x=0}^{L} k(x)dx = \frac{1}{L}\int_{x=0}^{L}\left(a + bx^{3/2}\right)dx = a + \frac{2}{5}bL^{3/2}$$

$$= 20\frac{\text{W}}{\text{m}\cdot\text{K}} + \frac{2}{5}\times 7070\frac{\text{W}}{\text{m}^{5/2}\cdot\text{K}}\times(0.02\text{m})^{3/2} = 28\frac{\text{W}}{\text{m}\cdot\text{K}}$$ <

Using this value, the heat rate per unit length is

$$q' = \bar{k}L(\Delta T)/L = 28\frac{\text{W}}{\text{m}\cdot\text{K}}\times 50°\text{C} = 1400\text{W/m}$$ <

(b) The nodal network is shown below. Note that the heat transfer is one-dimensional.



For any control surface, Eq. 4.46 may be combined with Fourier's law and written as

$$q'_{(m-1)-m} = \frac{T_{m-1} - T_m}{R'_{tot}} = \frac{T_{m-1} - T_m}{\left(\dfrac{\Delta x/2}{Lk_{m-1}} + \dfrac{\Delta x/2}{Lk_m}\right)} \qquad (1)$$

where the thermal conductivities, k_{m-1} and k_m are evaluated at the left (m – 1) and right (m) nodes, respectively. At steady state, the heat rate per unit length is constant. Hence, we may write Eqn. (1) for each pair of nodal points from m = 1 to m = 11 using Δx = 2 mm. The resulting heat rate per unit length is found by solving the 11 simultaneous equations for q′ and the temperatures T_m for $2 \le m \le 10$ yielding

q' = 1339 W/m <

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The predicted heat rate pure unit length is smaller than that of part (a).

(c) When the applied boundary conditions are changed to those of case 2, we may simply evaluate the heat transfer from the hot surface to the cool surface by evaluating the heat transfer in 11 different lanes and summing the results. For the interior lanes the width is Δx resulting in

$$q_m^{'} = \frac{(\Delta T)k_m}{L}\Delta x \quad \text{for } 2 \le m \le 10$$

where ΔT is the overall temperature difference across the domain. The thermal conductivities are evaluated at the locations of the nodes. For the lanes adjacent to the adiabatic boundaries,

$$q_1^{'} = k_1 \Delta T(\Delta x/2)/L \quad \text{and} \quad q_{11}^{'} k_{11} \Delta T(\Delta x/2)/L$$

and we evaluate the thermal conductivities k_1 and k_{11} at the nodal points, x = 0 and 20 mm, respectively. The heat rate per unit length of the object is

$$q' = \sum_{m=1}^{11} q_m^{'} \quad \text{or} \quad q' = 1401 \text{ W/m}$$ <

The heat rate per unit length is nearly identical to that of part (a).

COMMENTS: (1) The agreement between the results of parts (a) and (c) is expected since $q' = \int_{x=0}^{L} q''(x)dx = \int_{x=0}^{L} \left(k(x)\Delta T/L\right)dx = \bar{k}\left(L/L\right)\Delta T$. The minor difference is due to the evaluation of the thermal conductivity for each lane at the nodal point. The answers would become exactly the same as the spatial resolution of the numerical solution is increased. (2) In part (b) heat transfer is in the x-direction, the same direction in which thermal conductivity varies. This reduces heat transfer rates relative to the value calculated in parts (a) and (c). This is because the resistance expressed in Eq. (1) is composed of two values in series. The total resistance will be dominated by the higher of the two individual resistances. (3) Temperatures calculated for case 1 and heat rates in each lane for case 2 are shown in the table below.

Node or Lane	Temperature, °C (case 1)	Heat rate per unit length, W/m (case 2)
1	100.00	50
2	93.41	103.2
3	87.09	108.9
4	81.14	116.4
5	75.60	125.3
6	70.45	135.3
7	65.69	146.5
8	61.30	158.6
9	57.24	171.5
10	53.48	185.4
11	50.00	99.99
		1401