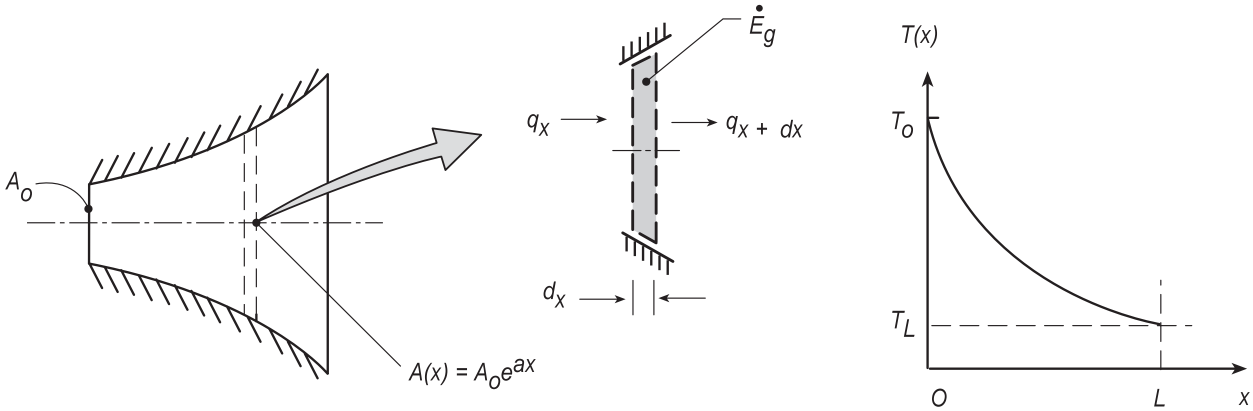
PROBLEM 2.16

KNOWN: A rod of constant thermal conductivity k and variable cross-sectional area $A_x(x) = A_o e^{ax}$ where A_o and a are constants.

FIND: (a) Expression for the conduction heat rate, $q_x(x)$; use this expression to determine the temperature distribution, T(x); and sketch of the temperature distribution, (b) Considering the presence of volumetric heat generation rate, $\dot{q} = \dot{q}_o \exp(-ax)$, obtain an expression for $q_x(x)$ when the left face, x = 0, is well insulated.

SCHEMATIC:



ASSUMPTIONS: (1) One-dimensional conduction in the rod, (2) Constant properties, (3) Steady-state conditions.

ANALYSIS: Perform an energy balance on the control volume, A(x)·dx,

$$\dot{E}_{in} - \dot{E}_{out} + \dot{E}_g = 0$$

$$q_x - q_{x+dx} + \dot{q} \cdot A(x) \cdot dx = 0$$

The conduction heat rate terms can be expressed as a Taylor series and substituting expressions for $\dot{q}$ and A(x),

$$-\frac{d}{dx}(q_x) + \dot{q}_o \exp(-ax) \cdot A_o \exp(ax) = 0 \tag{1}$$

$$q_x = -k \cdot A(x) \frac{dT}{dx} \tag{2}$$

(a) With no internal generation, $\dot{q}_o = 0$, and from Eq. (1) find

$$-\frac{d}{dx}(q_x) = 0$$ <

indicating that the heat rate is constant with x. By combining Eqs. (1) and (2)

$$-\frac{d}{dx}\left(-k \cdot A(x) \frac{dT}{dx}\right) = 0 \qquad \text{or} \qquad A(x) \cdot \frac{dT}{dx} = C_1 \tag{3}$$ <

Continued...

PROBLEM 2.16 (Cont.)

That is, the product of the cross-sectional area and the temperature gradient is a constant, independent of x. Hence, with T(0) > T(L), the temperature distribution is exponential, and as shown in the sketch above. Separating variables and integrating Eq. (3), the general form for the temperature distribution can be determined,

$$A_o \exp(ax) \cdot \frac{dT}{dx} = C_1$$

$$dT = C_1 A_o^{-1} \exp(-ax)\, dx$$

$$T(x) = -C_1 A_o a \exp(-ax) + C_2$$ <

We could use the two temperature boundary conditions, $T_o = T(0)$ and $T_L = T(L)$, to evaluate C_1 and C_2 and, hence, obtain the temperature distribution in terms of T_o and T_L.

(b) With the internal generation, from Eq. (1),

$$-\frac{d}{dx}(q_x) + \dot{q}_o A_o = 0 \qquad \text{or} \qquad q_x = \dot{q}_o A_o x$$ <

That is, the heat rate increases linearly with x.

COMMENTS: In part (b), you could determine the temperature distribution using Fourier's law and knowledge of the heat rate dependence upon the x-coordinate. Give it a try!