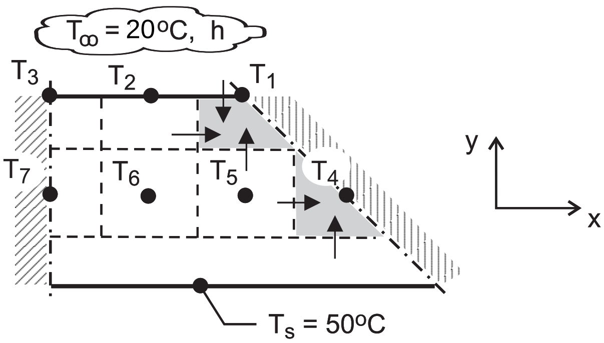
PROBLEM 4.58

KNOWN: Outer surface temperature, inner convection conditions, dimensions and thermal conductivity of a heat sink.

FIND: Nodal temperatures and heat rate per unit length.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state, (2) Two-dimensional conduction, (3) Uniform outer surface temperature, (4) Constant thermal conductivity.

ANALYSIS: (a) To determine the heat rate, the nodal temperatures must first be computed from the corresponding finite-difference equations. From an energy balance for node 1,

$$h(\Delta x/2\cdot 1)(T_\infty - T_1) + k(\Delta y/2\cdot 1)\frac{T_2 - T_1}{\Delta x} + k(\Delta x\cdot 1)\frac{T_5 - T_1}{\Delta y} = 0$$

$$-\left(3 + \frac{h\Delta x}{k}\right)T_1 + T_2 + 2T_5 + \frac{h\Delta x}{k}T_\infty = 0 \qquad (1)$$

With nodes 2 and 3 corresponding to Case 3 of Table 4.2,

$$T_1 - 2\left(\frac{h\Delta x}{k} + 2\right)T_2 + T_3 + 2T_6 + \frac{2h\Delta x}{k}T_\infty = 0 \qquad (2)$$

$$T_2 - \left(\frac{h\Delta x}{k} + 2\right)T_3 + T_7 + \frac{h\Delta x}{k}T_\infty = 0 \qquad (3)$$

where the symmetry condition is invoked for node 3. Applying an energy balance to node 4, we obtain

$$-2T_4 + T_5 + T_s = 0 \qquad (4)$$

The interior nodes 5, 6 and 7 correspond to Case 1 of Table 4.2. Hence,

$$T_1 + T_4 - 4T_5 + T_6 + T_s = 0 \qquad (5)$$
$$T_2 + T_5 - 4T_6 + T_7 + T_s = 0 \qquad (6)$$
$$T_3 + 2T_6 - 4T_7 + T_s = 0 \qquad (7)$$

where the symmetry condition is invoked for node 7. With $T_s = 50°C$, $T_\infty = 20°C$, and $h\Delta x/k = 5000\ W/m^2\cdot K(0.005m)/240\ W/m\cdot K = 0.1042$, the solution to Eqs. (1) – (7) yields

$$T_1 = 46.61°C,\ T_2 = 45.67°C,\ T_3 = 45.44°C,\ T_4 = 49.23°C$$

$$T_5 = 48.46°C,\ T_6 = 48.00°C,\ T_7 = 47.86°C$$ <

Continued …

PROBLEM 4.58 (Cont.)

The heat rate per unit length of channel may be evaluated by computing convection heat transfer from the inner surface. That is,

$$q' = 8h\left[\Delta x/2\left(T_1 - T_\infty\right) + \Delta x\left(T_2 - T_\infty\right) + \Delta x/2\left(T_3 - T_\infty\right)\right]$$

$$q' = 8\times 5000\,\mathrm{W/m^2\cdot K}\left[0.0025\mathrm{m}\left(46.61-20\right)°\mathrm{C} + 0.005\mathrm{m}\left(45.67-20\right)°\mathrm{C}\right.$$

$$\left. +0.0025\mathrm{m}\left(45.44-20\right)°\mathrm{C}\right] = 10{,}340\ \mathrm{W/m}$$ <

(b) Since $h = 5000\ \mathrm{W/m^2\cdot K}$ is at the high end of what can be achieved through forced convection, we consider the effect of reducing h. Representative results are as follows

$h\left(\mathrm{W/m^2\cdot K}\right)$	$T_1\,(°\mathrm{C})$	$T_2\,(°\mathrm{C})$	$T_3\,(°\mathrm{C})$	$T_4\,(°\mathrm{C})$	$T_5\,(°\mathrm{C})$	$T_6\,(°\mathrm{C})$	$T_7\,(°\mathrm{C})$	$q'\,(\mathrm{W/m})$
200	49.84	49.80	49.79	49.96	49.93	49.91	49.90	477
1000	49.24	49.02	48.97	49.83	49.65	49.55	49.52	2325
2000	48.53	48.11	48.00	49.66	49.33	49.13	49.06	4510
5000	46.61	45.67	45.44	49.23	48.46	48.00	47.86	10,340

There are two resistances to heat transfer between the outer surface of the heat sink and the fluid, that due to conduction in the heat sink, $R_{cond(2D)}$, and that due to convection from its inner surface to the fluid, R_{conv}. With decreasing h, the corresponding increase in R_{conv} reduces heat flow and increases the uniformity of the temperature field in the heat sink. The nearly 5-fold reduction in q' corresponding to the 5-fold reduction in h from 1000 to 200 $\mathrm{W/m^2\cdot K}$ indicates that the convection resistance is dominant $\left(R_{conv} \gg R_{cond(2D)}\right)$.

COMMENTS: To check our finite-difference solution, we could assess its consistency with conservation of energy requirements. For example, an energy balance performed at the inner surface requires a balance between convection from the surface and conduction to the surface, which may be expressed as

$$q' = k\left(\Delta x\cdot 1\right)\frac{\left(T_5 - T_1\right)}{\Delta y} + k\left(\Delta x\cdot 1\right)\frac{T_6 - T_2}{\Delta y} + k\left(\Delta x/2\cdot 1\right)\frac{T_7 - T_3}{\Delta y}$$

Substituting the temperatures corresponding to $h = 5000\ \mathrm{W/m^2\cdot K}$, the expression yields $q' = 10{,}340\,\mathrm{W/m}$, and, as it must be, conservation of energy is precisely satisfied. Results of the analysis may also be checked by using the expression $q' = \left(T_s - T_\infty\right)/\left(R'_{cond(2D)} + R'_{conv}\right)$, where, for $h = 5000\ \mathrm{W/m^2\cdot K}$, $R'_{conv} = \left(1/4hw\right) = 2.5\times 10^{-3}\mathrm{m\cdot K/W}$, and from Eq. (4.27) and Case 11 of Table 4.1, $R'_{cond} = \left[0.930\ \ln\left(W/w\right) - 0.05\right]/2\pi k = 3.94\times 10^{-4}\,\mathrm{m\cdot K/W}$. Hence, $q' = \left(50-20\right)°\mathrm{C}/\left(2.5\times 10^{-3} + 3.94\times 10^{-4}\right)\mathrm{m\cdot K/W} = 10{,}370\ \mathrm{W/m}$, and the agreement with the finite-difference solution is excellent. Note that, even for $h = 5000\ \mathrm{W/m^2\cdot K}$, $R'_{conv} \gg R'_{cond(2D)}$.