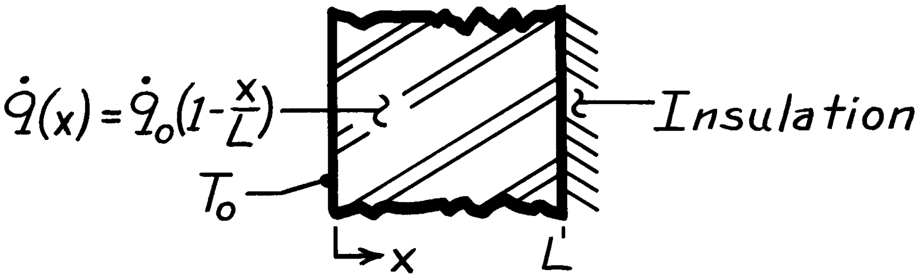
PROBLEM 3.93

KNOWN: Plane wall with prescribed nonuniform volumetric generation having one boundary insulated and the other isothermal.

FIND: Temperature distribution, T(x), in terms of x, L, k, $\dot{q}_o$ and T_o.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) One-dimensional conduction in x-direction, (3) Constant properties.

ANALYSIS: The appropriate form the heat diffusion equation is

$$\frac{d}{dx}\left[\frac{dT}{dx}\right]+\frac{\dot{q}}{k}=0.$$

Noting that $\dot{q}=\dot{q}(x)=\dot{q}_o\ (1-x/L)$, substitute for $\dot{q}(x)$ into the above equation, separate variables and then integrate,

$$d\left[\frac{dT}{dx}\right]=-\frac{\dot{q}_o}{k}\left[1-\frac{x}{L}\right]dx \qquad \frac{dT}{dx}=-\frac{\dot{q}_o}{k}\left[x-\frac{x^2}{2L}\right]+C_1.$$

Separate variables and integrate again to obtain the general form of the temperature distribution in the wall,

$$dT=-\frac{\dot{q}_o}{k}\left[x-\frac{x^2}{2L}\right]dx+C_1dx \qquad T(x)=-\frac{\dot{q}_o}{k}\left[\frac{x^2}{2}-\frac{x^3}{6L}\right]+C_1x+C_2.$$

Identify the boundary conditions at x = 0 and x = L to evaluate C_1 and C_2. At x = 0,

$$T(0)=T_o=-\frac{\dot{q}_o}{k}(0-0)+C_1\cdot 0+C_2 \qquad \text{hence, } C_2=T_o$$

At x = L,

$$\left.\frac{dT}{dx}\right]_{x=L}=0=-\frac{\dot{q}_o}{k}\left[L-\frac{L^2}{2L}\right]+C_1 \qquad \text{hence, } C_1=\frac{\dot{q}_oL}{2k}$$

The temperature distribution is

$$T(x)=-\frac{\dot{q}_o}{k}\left[\frac{x^2}{2}-\frac{x^3}{6L}\right]+\frac{\dot{q}_oL}{2k}x+T_o.$$ <

COMMENTS: It is good practice to test the final result for satisfying BCs. The heat flux at x = 0 can be found using Fourier's law or from an overall energy balance

$$\dot{E}_{out}=\dot{E}_g=\int_0^L \dot{q}dV \qquad \text{to obtain} \qquad q''_{out}=\dot{q}_oL/2.$$