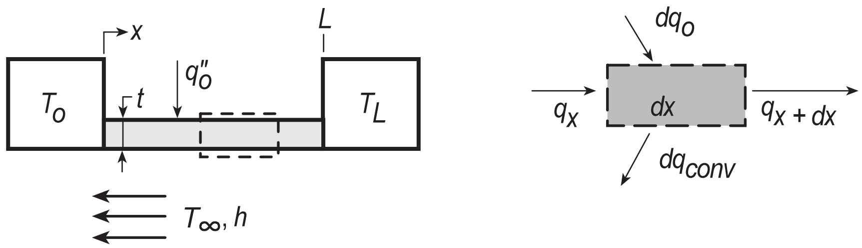
PROBLEM 3.113

KNOWN: Dimensions and surface conditions of a plate thermally joined at its ends to heat sinks at different temperatures. Heat flux into top of plate. Convection conditions beneath plate.

FIND: (a) Differential equation which determines temperature distribution in plate, (b) Temperature distribution and an expression for the heat rate from the plate to the sinks, and (c) Compute and plot temperature distribution and heat rates corresponding to changes in different parameters.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) One-dimensional conduction in x (W,L >> t), (3) Constant properties, (4) Uniform surface heat flux and convection coefficient, (5) Negligible contact resistance.

ANALYSIS: (a) Applying conservation of energy to the differential control volume

$$q_x + dq_o = q_{x+dx} + dq_{conv}$$

where

$$q_{x+dx} = q_x + \left(dq_x/dx\right)dx \qquad\qquad dq_{conv} = h\left(T - T_\infty\right)\left(W \cdot dx\right)$$

Hence,

$$q_x + q''_o\left(W \cdot dx\right) = q_x + \left(dq_x/dx\right)dx + h\left(T - T_\infty\right)\left(W \cdot dx\right) \qquad\qquad \frac{dq_x}{dx} + hW\left(T - T_\infty\right) = q''_o W\,.$$

Using Fourier's law, $q_x = -k\left(t \cdot W\right)dT/dx$,

$$-ktW\frac{d^2T}{dx^2} + hW\left(T - T_\infty\right) = q''_o W \qquad\qquad \frac{d^2T}{dx^2} - \frac{h}{kt}\left(T - T_\infty\right) + \frac{q''_o}{kt} = 0\,.$$ <

(b) Introducing $\theta \equiv T - T_\infty$, the differential equation becomes

$$\frac{d^2\theta}{dx^2} - \frac{h}{kt}\theta + \frac{q''_o}{kt} = 0\,.$$

This differential equation is of second order with constant coefficients and a source term. With $\lambda^2 \equiv h/kt$ and $S \equiv q''_o/kt$, it follows that the general solution is of the form

$$\theta = C_1 e^{+\lambda x} + C_2 e^{-\lambda x} + S/\lambda^2\,. \tag{1}$$

Appropriate boundary conditions are: $\quad \theta(0) = T_o - T_\infty \equiv \theta_o \qquad\qquad \theta(L) = T_L - T_\infty \equiv \theta_L \qquad$ (2,3)

Substituting the boundary conditions, Eqs. (2,3) into the general solution, Eq. (1),

$$\theta_o = C_1 e^0 + C_2 e^0 + S/\lambda^2 \qquad\qquad \theta_L = C_1 e^{+\lambda L} + C_2 e^{-\lambda L} + S/\lambda^2 \tag{4,5}$$

To solve for C_2, multiply Eq. (4) by $-e^{+\lambda L}$ and add the result to Eq. (5),

$$-\theta_o e^{+\lambda L} + \theta_L = C_2\left(-e^{+\lambda L} + e^{-\lambda L}\right) + S/\lambda^2\left(-e^{+\lambda L} + 1\right)$$

$$C_2 = \left[\left(\theta_L - \theta_o e^{+\lambda L}\right) - S/\lambda^2\left(-e^{+\lambda L} + 1\right)\right] \Big/ \left(-e^{+\lambda L} + e^{-\lambda L}\right) \tag{6}$$

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PROBLEM 3.113 (Cont.)

Substituting for C_2 from Eq. (6) into Eq. (4), find

$$C_1 = \theta_o - \left\{ \left[\left(\theta_L - \theta_o e^{+\lambda L} \right) - S/\lambda^2 \left(-e^{+\lambda L} + 1 \right) \right] \Big/ \left(-e^{+\lambda L} + e^{-\lambda L} \right) \right\} - S/\lambda^2 \qquad (7)$$

Using C_1 and C_2 from Eqs. (6,7) and Eq. (1), the temperature distribution can be expressed as

$$\theta(x) = \left[e^{+\lambda x} - \frac{\sinh(\lambda x)}{\sinh(\lambda L)} e^{+\lambda L} \right] \theta_o + \frac{\sinh(\lambda x)}{\sinh(\lambda L)} \theta_L + \left[-\left(1 - e^{+\lambda L}\right) \frac{\sinh(\lambda x)}{\sinh(\lambda L)} + \left(1 - e^{+\lambda x}\right) \right] \frac{S}{\lambda^2} \qquad (8)$$ <

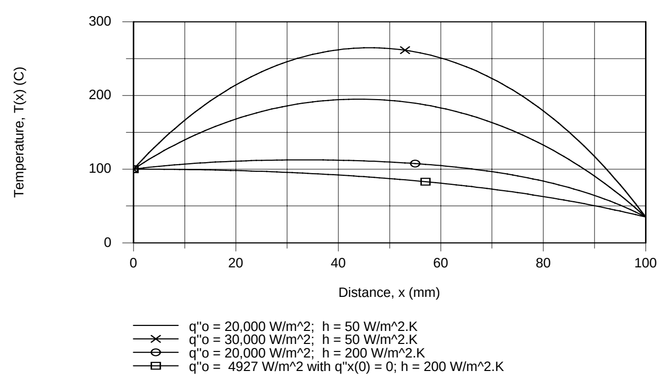
The heat rate from the plate is $q_p = -q_x(0) + q_x(L)$ and using Fourier's law, the conduction heat rates, with $A_c = W \cdot t$, are

$$q_x(0) = -kA_c \left.\frac{d\theta}{dx}\right)_{x=0} = -kA_c \left\{ \left[\lambda e^0 - \frac{e^{\lambda L}}{\sinh(\lambda L)} \lambda \right] \theta_o + \frac{\lambda}{\sinh(\lambda L)} \theta_L + \left[-\frac{1 - e^{+\lambda L}}{\sinh(\lambda L)} \lambda - \lambda \right] \frac{S}{\lambda^2} \right\}$$ <

$$q_x(L) = -kA_c \left.\frac{d\theta}{dx}\right)_{x=L} = -kA_c \left\{ \left[\lambda e^{\lambda L} - \frac{e^{\lambda L}}{\sinh(\lambda L)} \lambda \cosh(\lambda L) \right] \theta_o + \frac{\lambda \cosh(\lambda L)}{\sinh(\lambda L)} \theta_L + \left[-\frac{1 - e^{+\lambda L}}{\sinh(\lambda L)} \lambda \cosh(\lambda L) - \lambda e^{+\lambda L} \right] \frac{S}{\lambda^2} \right\}$$ <

(c) For the prescribed base-case conditions listed below, the temperature distribution (solid line) is shown in the accompanying plot. As expected, the maximum temperature does not occur at the midpoint, but slightly toward the x-origin. The sink heat rates are

$$q''_x(0) = -17.22\,\text{W} \qquad\qquad q''_x(L) = 23.62\,\text{W}$$ <



The additional temperature distributions on the plot correspond to changes in the following parameters, with all the remaining parameters unchanged: (i) q''_o = 30,000 W/m^2, (ii) h = 200 W/m$^2\cdot$K, (iii) the value of q''_o for which $q''_x(0) = 0$ with h = 200 W/m$^2\cdot$K. The condition for the last curve is q''_o = 4927 W/m^2 for which the temperature gradient at x = 0 is zero.

Base case conditions are: q''_o = 20,000 W/m^2, T_o = 100°C, T_L = 35°C, T_∞ = 25°C, k = 25 W/m·K, h = 50 W/m$^2\cdot$K, L = 100 mm, t = 5 mm, W = 30 mm.