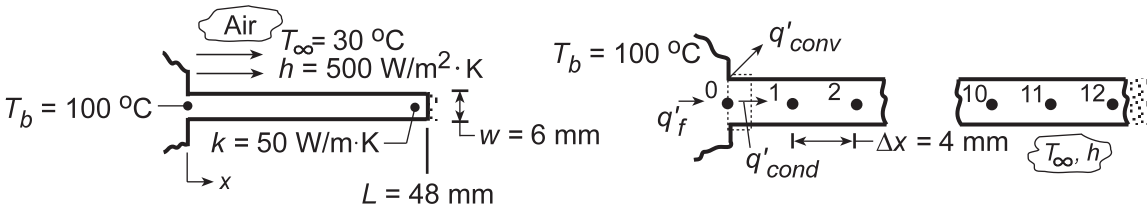
PROBLEM 4.78

KNOWN: Straight fin of uniform cross section with insulated end.

FIND: (a) Temperature distribution using finite-difference method and validity of assuming one-dimensional heat transfer, (b) Fin heat transfer rate and comparison with analytical solution, Eq. 3.81, (c) Effect of convection coefficient on fin temperature distribution and heat rate.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) One-dimensional conduction in fin, (3) Constant properties, (4) Uniform film coefficient.

ANALYSIS: (a) From the analysis of Problem 4.50, the finite-difference equations for the nodal arrangement can be directly written. For the nodal spacing Δx = 4 mm, there will be 12 nodes. With ℓ >> w representing the distance normal to the page,

$$\frac{hP}{kA_c}\cdot\Delta x^2 \approx \frac{h\cdot 2\ell}{k\cdot\ell\cdot w}\Delta x^2 = \frac{h\cdot 2}{kw}\Delta x^2 = \frac{500\,\mathrm{W/m^2\cdot K}\times 2}{50\,\mathrm{W/m\cdot K}\times 6\times 10^{-3}\,\mathrm{m}}\left(4\times 10^{-3}\mathrm{mm}\right) = 0.0533$$

Node 1: $100 + T_2 + 0.0533\times 30 - (2 + 0.0533)T_1 = 0$ or $-2.053T_1 + T_2 = -101.6$

Node n: $T_{n+1} + T_{n-1} + 1.60 - 2.0533T_n = 0$ or $T_{n-1} - 2.053T_n + T_{n-1} = -1.60$

Node 12: $T_{11} + (0.0533/2)30 - (0.0533/2 + 1)T_{12} = 0$ or $T_{11} - 1.0267T_{12} = -0.800$

Using matrix notation, Eq. 4.48, where [A] [T] = [C], the A-matrix is tridiagonal and only the non-zero terms are shown below. A matrix inversion routine was used to obtain [T].

Tridiagonal Matrix A						*Column Matrices*		
Nonzero Terms				Values		Node	C	T
	$a_{1,1}$	$a_{1,2}$		-2.053	1	1	-101.6	85.8
$a_{2,1}$	$a_{2,2}$	$a_{2,3}$	1	-2.053	1	2	-1.6	74.5
$a_{3,2}$	$a_{3,3}$	$a_{3,4}$	1	-2.053	1	3	-1.6	65.6
$a_{4,3}$	$a_{4,4}$	$a_{4,5}$	1	-2.053	1	4	-1.6	58.6
$a_{5,4}$	$a_{5,5}$	$a_{5,6}$	1	-2.053	1	5	-1.6	53.1
$a_{6,5}$	$a_{6,6}$	$a_{6,7}$	1	-2.053	1	6	-1.6	48.8
$a_{7,6}$	$a_{7,7}$	$a_{7,8}$	1	-2.053	1	7	-1.6	45.5
$a_{8,7}$	$a_{8,8}$	$a_{8,9}$	1	-2.053	1	8	-1.6	43.0
$a_{9,8}$	$a_{9,9}$	$a_{9,10}$	1	-2.053	1	9	-1.6	41.2
$a_{10,9}$	$a_{10,10}$	$a_{10,11}$	1	-2.053	1	10	-1.6	39.9
$a_{11,10}$	$a_{11,11}$	$a_{11,12}$	1	-2.053	1	11	-1.6	39.2
$a_{12,11}$	$a_{12,12}$	$a_{12,13}$	1	-1.027	1	12	-0.8	38.9

The assumption of one-dimensional heat conduction is justified when Bi ≡ h(w/2)/k < 0.1. Hence, with Bi = 500 W/m^2·K(3 × 10^{-3} m)/50 W/m·K = 0.03, the assumption is reasonable.

Continued...

PROBLEM 4.78 (Cont.)

(b) The fin heat rate can be most easily found from an energy balance on the control volume about Node 0,

$$q'_f = q'_1 + q'_{conv} = k \cdot w \frac{T_0 - T_1}{\Delta x} + h\left(2\frac{\Delta x}{2}\right)(T_0 - T_\infty)$$

$$q'_f = 50\,W/m\cdot K\left(6\times10^{-3}m\right)\frac{(100-85.8)^\circ C}{4\times10^{-3}m} + 500\,W/m^2\cdot K\left(2\cdot\frac{4\times10^{-3}m}{2}\right)(100-30)^\circ C$$

$$q'_f = (1065+140)\,W/m = 1205\,W/m\,.$$ <

From Eq. 3.81, the fin heat rate is

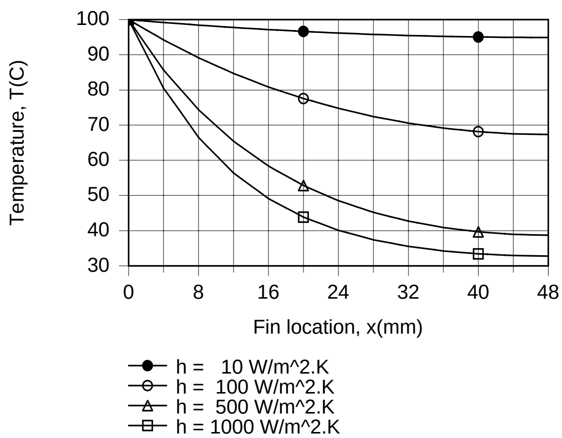
$$q = \left(hPkA_c\right)^{1/2}\cdot\theta_b\cdot\tanh mL\,.$$

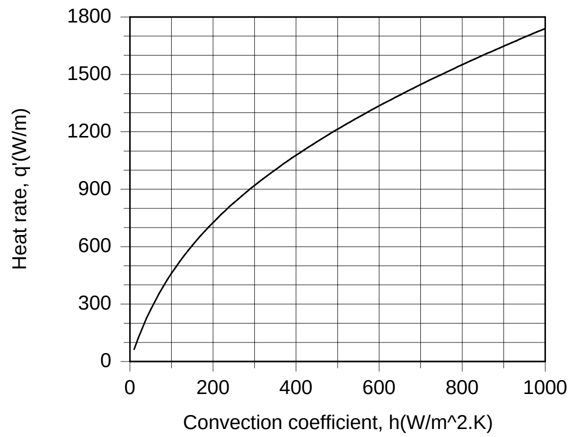
Substituting numerical values with $P = 2(w + \ell) \approx 2\ell$ and $A_c = w\cdot\ell$, $m = (hP/kA_c)^{1/2} = 57.74\ m^{-1}$ and $M = (hPkA_c)^{1/2} = 17.32\,\ell$ W/K. Hence, with $\theta_b = 70°C$,

$$q' = 17.32\,W/K\times70\,K\times\tanh(57.44\times0.048) = 1203\,W/m$$

and the finite-difference result agrees very well with the exact (analytical) solution.

(c) Using the IHT *Finite-Difference Equations Tool Pad* for *1D, SS* conditions, the fin temperature distribution and heat rate were computed for h = 10, 100, 500 and 1000 $W/m^2\cdot K$. Results are plotted as follows.





The temperature distributions were obtained by first creating a *Lookup Table* consisting of 4 rows of nodal temperatures corresponding to the 4 values of h and then using the *LOOKUPVAL2* interpolating function with the *Explore* feature of the IHT menu. Specifically, the function T_EVAL = LOOKUPVAL2(t0467, h, x) was entered into the workspace, where t0467 is the file name given to the Lookup Table. For each value of h, *Explore* was used to compute T(x), thereby generating 4 data sets which were placed in the *Browser* and used to generate the plots. The variation of q' with h was simply generated by using the *Explore* feature to solve the finite-difference model equations for values of h incremented by 10 from 10 to 1000 $W/m^2\cdot K$.

Although q'_f increases with increasing h, the effect of changes in h becomes less pronounced. This trend is a consequence of the reduction in fin temperatures, and hence the fin efficiency, with increasing h. For $10 \le h \le 1000\ W/m^2\cdot K$, $0.95 \ge \eta_f \ge 0.24$. Note the nearly isothermal fin for h = 10 $W/m^2\cdot K$ and the pronounced temperature decay for h = 1000 $W/m^2\cdot K$.