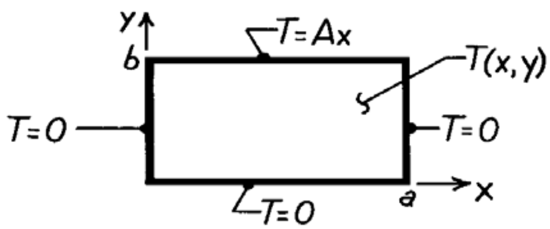
PROBLEM 4.4

KNOWN: Rectangular plate subjected to prescribed boundary conditions.

FIND: Steady-state temperature distribution.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state, 2-D conduction, (2) Constant properties.

ANALYSIS: The solution follows the method of Section 4.2. The product solution is

$$T(x,y) = X(x)\cdot Y(y) = (C_1\cos\lambda x + C_2\sin\lambda x)\left(C_3 e^{-\lambda y} + C_4 e^{+\lambda y}\right)$$

and the boundary conditions are: T(0,y) = 0, T(a,y) = 0, T(x,0) = 0, T(x.b) = Ax. Applying BC#1, T(0,y) = 0, find $C_1 = 0$. Applying BC#2, T(a,y) = 0, find that $\lambda = n\pi/a$ with n = 1,2,…. Applying BC#3, T(x,0) = 0, find that $C_3 = -C_4$. Hence, the product solution is

$$T(x,y) = X(x)\cdot Y(y) = C_2 C_4\ \sin\left[\frac{n\pi}{a}x\right]\ \left(e^{+\lambda y} - e^{-\lambda y}\right).$$

Combining constants and using superposition, find

$$T(x,y) = \sum_{n=1}^{\infty} C_n\ \sin\left[\frac{n\pi x}{a}\right] \sinh\left[\frac{n\pi y}{a}\right].$$

To evaluate C_n and satisfy BC#4, use orthogonal functions with Equation 4.16 to find

$$C_n = \int_0^a Ax\cdot\sin\left[\frac{n\pi x}{a}\right]\cdot dx/\sinh\left[\frac{n\pi b}{a}\right]\int_0^a \sin^2\left[\frac{n\pi x}{a}\right] dx,$$

noting that y = b. The numerator, denominator and C_n, respectively, are:

$$A\int_0^a x\cdot\sin\frac{n\pi x}{a}\cdot dx = A\left[\left[\frac{a}{n\pi}\right]^2 \sin\left[\frac{n\pi x}{a}\right] - \frac{ax}{n\pi}\cos\left[\frac{n\pi x}{a}\right]\right]_0^a = \frac{Aa^2}{n\pi}\left[-\cos(n\pi)\right] = \frac{Aa^2}{n\pi}(-1)^{n+1},$$

$$\sinh\left[\frac{n\pi b}{a}\right]\int_0^a \sin^2\frac{n\pi x}{a}\cdot dx = \sinh\left[\frac{n\pi b}{a}\right]\left[\frac{1}{2}x - \frac{a}{4n\pi}\sin\left[\frac{2n\pi x}{a}\right]\right]_0^a = \frac{a}{2}\cdot\sinh\left[\frac{n\pi b}{a}\right],$$

$$C_n = \frac{Aa^2}{n\pi}(-1)^{n+1}/\frac{a}{2}\sinh\left[\frac{n\pi b}{a}\right] = 2Aa\ (-1)^{n+1}/n\pi\ \sinh\left[\frac{n\pi b}{a}\right].$$

Hence, the temperature distribution is

$$T(x,y) = \frac{2\ Aa}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\cdot\sin\left[\frac{n\pi x}{a}\right]\frac{\sinh\left[\frac{n\pi y}{a}\right]}{\sinh\left[\frac{n\pi b}{a}\right]}.$$ <