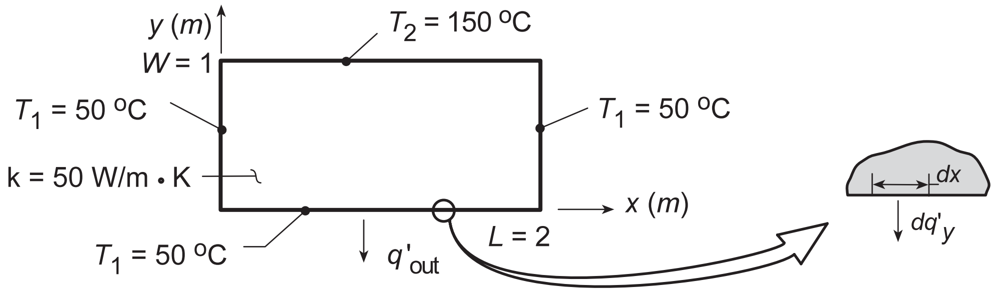
PROBLEM 4.3

KNOWN: Temperature distribution in the two-dimensional rectangular plate of Problem 4.2.

FIND: Expression for the heat rate per unit thickness from the lower surface ($0 \le x \le 2$, 0) and result based on first five non-zero terms of the infinite series.

SCHEMATIC:



ASSUMPTIONS: (1) Two-dimensional, steady-state conduction, (2) Constant properties.

ANALYSIS: The heat rate per unit thickness *from the plate* along the lower surface is

$$q'_{out} = -\int_{x=0}^{x=2} dq'_y(x,0) = -\int_{x=0}^{x=2} -k\left.\frac{\partial T}{\partial y}\right|_{y=0} dx = k\left(T_2 - T_1\right)\int_{x=0}^{x=2} \left.\frac{\partial \theta}{\partial y}\right|_{y=0} dx \qquad (1)$$

where from the solution to Problem 4.2,

$$\theta \equiv \frac{T - T_1}{T_2 - T_1} = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}+1}{n}\sin\left(\frac{n\pi x}{L}\right)\frac{\sinh(n\pi y/L)}{\sinh(n\pi W/L)}. \qquad (2)$$

Evaluate the gradient of θ from Eq. (2) and substitute into Eq. (1) to obtain

$$q'_{out} = k\left(T_2 - T_1\right)\int_{x=0}^{x=2}\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}+1}{n}\sin\left(\frac{n\pi x}{L}\right)\left.\frac{(n\pi/L)\cosh(n\pi y/L)}{\sinh(n\pi W/L)}\right|_{y=0} dx$$

$$q'_{out} = k\left(T_2 - T_1\right)\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}+1}{n}\frac{1}{\sinh(n\pi W/L)}\left[-\cos\left(\frac{n\pi x}{L}\right)\Big|_{x=0}^{2}\right]$$

$$q'_{out} = k\left(T_2 - T_1\right)\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}+1}{n}\frac{1}{\sinh(n\pi/L)}\left[1-\cos(n\pi)\right]$$ <

To evaluate the first five, non-zero terms, recognize that since cos(nπ) = 1 for n = 2, 4, 6 ..., only the n-odd terms will be non-zero. Hence,

Continued …

PROBLEM 4.3 (Cont.)

$$q'_{out} = 50\,\mathrm{W/m\cdot K}(150-50)^\circ\mathrm{C}\,\frac{2}{\pi}\left\{\frac{(-1)^2+1}{1}\cdot\frac{1}{\sinh(\pi/2)}(2)+\frac{(-1)^4+1}{3}\cdot\frac{1}{\sinh(3\pi/2)}\cdot(2)\right.$$

$$\left.+\frac{(-1)^6+1}{5}\cdot\frac{1}{\sinh(5\pi/2)}(2)+\frac{(-1)^8+1}{7}\cdot\frac{1}{\sinh(7\pi/2)}(2)+\frac{(-1)^{10}+1}{9}\cdot\frac{1}{\sinh(9\pi/2)}(2)\right\}$$

$$q'_{out} = 3.183\,\mathrm{kW/m}\left[1.738+0.024+0.00062+(...)\right] = 5.611\,\mathrm{kW/m}$$ <

COMMENTS: If the foregoing procedure were used to evaluate the heat rate into the upper surface, $q'_{in} = -\int_{x=0}^{x=2} dq'_y(x,W)$, it would follow that

$$q'_{in} = k(T_2 - T_1)\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}+1}{n}\coth(n\pi/2)\left[1-\cos(n\pi)\right]$$

However, with $\coth(n\pi/2) \geq 1$, irrespective of the value of n, and with $\sum_{n=1}^{\infty}\left[(-1)^{n+1}+1\right]/n$ being a divergent series, the complete series does not converge and $q'_{in} \to \infty$. This physically untenable condition results from the temperature discontinuities imposed at the upper left and right corners.