PROBLEM 2.36

KNOWN: Three-dimensional system – described by spherical coordinates (r,ϕ,θ) – experiences transient conduction and internal heat generation.

FIND: Heat diffusion equation.

SCHEMATIC: See Figure 2.13.

ASSUMPTIONS: (1) Homogeneous medium.

ANALYSIS: The differential control volume is $V = dr \cdot r\sin\theta d\phi \cdot r d\theta$, and the conduction terms are identified in Figure 2.13. Conservation of energy requires

$$q_r - q_{r+dr} + q_\phi - q_{\phi+d\phi} + q_\theta - q_{\theta+d\theta} + \dot{E}_g = \dot{E}_{st}. \qquad (1)$$

The generation and storage terms, both representing volumetric phenomena, are

$$\dot{E}_g = \dot{q}V = \dot{q}\left[dr \cdot r\sin\theta d\phi \cdot r d\theta\right] \qquad \dot{E}_{st} = \rho V c \frac{\partial T}{\partial t} = \rho\left[dr \cdot r\sin\theta d\phi \cdot r d\theta\right] c \frac{\partial T}{\partial t}. \qquad (2,3)$$

Using a Taylor series expansion, we can write

$$q_{r+dr} = q_r + \frac{\partial}{\partial r}(q_r)dr, \quad q_{\phi+d\phi} = q_\phi + \frac{\partial}{\partial\phi}(q_\phi)d\phi, \quad q_{\theta+d\theta} = q_\theta + \frac{\partial}{\partial\theta}(q_\theta)d\theta. \qquad (4,5,6)$$

From Fourier's law, the conduction heat rates have the following forms.

$$q_r = -kA_r \partial T/\partial r = -k\left[r\sin\theta d\phi \cdot r d\theta\right]\partial T/\partial r \qquad (7)$$

$$q_\phi = -kA_\phi \partial T/r\sin\theta\partial\phi = -k\left[dr \cdot r d\theta\right]\partial T/r\sin\theta\partial\phi \qquad (8)$$

$$q_\theta = -kA_\theta \partial T/r\partial\theta = -k\left[dr \cdot r\sin\theta d\phi\right]\partial T/r\partial\theta. \qquad (9)$$

Substituting Eqs. (2), (3) and (4), (5), (6) into Eq. (1), the energy balance becomes

$$-\frac{\partial}{\partial r}(q_r)dr - \frac{\partial}{\partial\phi}(q_\phi)d\phi - \frac{\partial}{\partial\theta}(q_\theta)d\theta + \dot{q}\left[dr \cdot r\sin\theta d\phi \cdot r d\theta\right] = \rho\left[dr \cdot r\sin\theta d\phi \cdot r d\theta\right] c\frac{\partial T}{\partial t} \qquad (10)$$

Substituting Eqs. (7), (8) and (9) for the conduction rates, find

$$-\frac{\partial}{\partial r}\left[-k\left[r\sin\theta d\phi \cdot r d\theta\right]\frac{\partial T}{\partial r}\right]dr - \frac{\partial}{\partial\phi}\left[-k\left[dr \cdot r d\theta\right]\frac{\partial T}{r\sin\theta\partial\phi}\right]d\phi$$

$$-\frac{\partial}{\partial\theta}\left[-k\left[dr \cdot r\sin\theta d\phi\right]\frac{\partial T}{r\partial\theta}\right]d\theta + \dot{q}\left[dr \cdot r\sin\theta d\phi \cdot r d\theta\right] = \rho\left[dr \cdot r\sin\theta d\phi \cdot r d\theta\right] c\frac{\partial T}{\partial t} \qquad (11)$$

Dividing Eq. (11) by the volume of the control volume, V, Eq. 2.29 is obtained.

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left[kr^2\frac{\partial T}{\partial r}\right] + \frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial\phi}\left[k\frac{\partial T}{\partial \phi}\right] + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left[k\sin\theta\frac{\partial T}{\partial\theta}\right] + \dot{q} = \rho c\frac{\partial T}{\partial t}. \qquad <$$

COMMENTS: Note how the temperature gradients in Eqs. (7) - (9) are formulated. The numerator is always ∂T while the denominator is the dimension of the control volume in the specified coordinate direction.