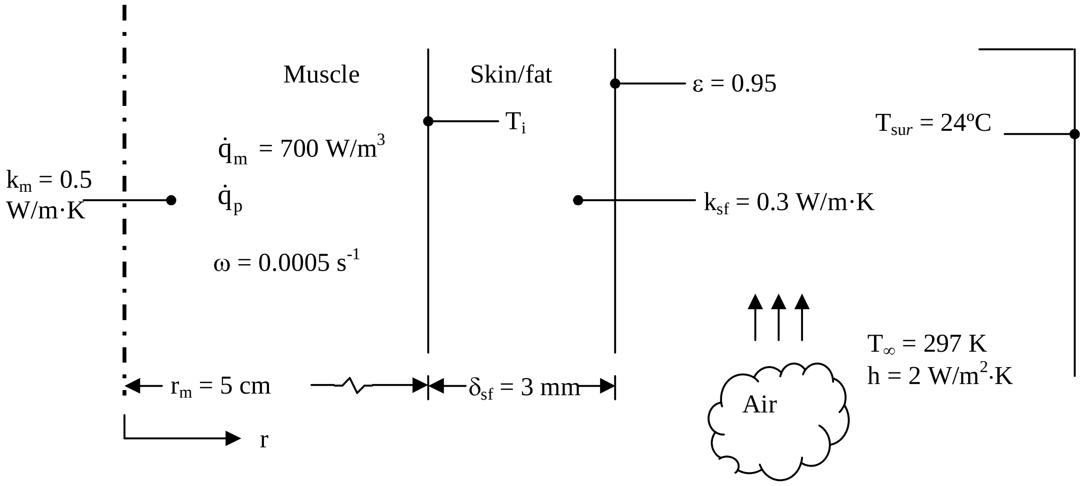
PROBLEM 3.167

KNOWN: Dimensions and thermal conductivities of a muscle layer and a skin/fat layer. Metabolic heat generation rate and perfusion rate within the muscle layer. Arterial temperature. Blood density and specific heat. Ambient conditions.

FIND: Heat loss rate from body and temperature at inner surface of the skin/fat layer.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) One-dimensional heat transfer through the muscle and skin/fat layers, (3) Metabolic heat generation rate, perfusion rate, arterial temperature, blood properties, and thermal conductivities are all uniform, (4) Radiation heat transfer coefficient is known from Example 1.6.

ANALYSIS:

(a) Conduction with heat generation is expressed in radial coordinates by Equation 3.54. With metabolic heat generation and perfusion, this becomes

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) + \frac{\dot{q}_m + \omega\rho_b c_b (T_a - T)}{k} = 0$$ <

The boundary conditions of symmetry at the centerline and specified temperature at the outer surface of the muscle are expressed as

$$\left.\frac{dT}{dr}\right|_{r=0} = 0, \quad T(r_1) = T_i$$ <

Defining an excess temperature, $\theta \equiv T - T_a - \dot{q}_m/\omega\rho_b c_b$, the differential equation becomes

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\theta}{dr}\right) - \tilde{m}^2\theta = 0$$

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where $\tilde{m}^2 = \omega\rho_b c_b/k$. The general solution to the differential equation is given in Section 3.6.4 as

$$\theta = c_1 I_0(\tilde{m}r) + c_2 K_0(\tilde{m}r)$$

Applying the boundary condition at r = 0 yields

$$\left.\frac{dT}{dr}\right|_{r=0} = \left.\frac{d\theta}{dr}\right|_{r=0} = c_1\tilde{m}I_1(0) - c_2\tilde{m}K_1(0) = 0$$

Since $K_1(0)$ is infinite, we must have $c_2 = 0$. Applying the specified temperature boundary condition at $r = r_1$ yields

$$T(r_1) = T_i,\ \theta(r_1) = T_i - T_a - \frac{\dot{q}_m}{\omega\rho_b c_b} \equiv \theta_i = c_1 I_0(\tilde{m}r_1)$$

Solving for c_1 we now have the complete solution for θ:

$$\theta = \theta_i \frac{I_0(\tilde{m}r)}{I_0(\tilde{m}r_1)} \qquad (1)$$ <

(b) The heat flux at the outer surface of the muscle is given by

$$q_i'' = -k_m \left.\frac{dT}{dr}\right|_{r=r_1} = -k_m \left.\frac{d\theta}{dr}\right|_{r=r_1} = -k_m\theta_i\tilde{m}\frac{I_1(\tilde{m}r_1)}{I_0(\tilde{m}r_1)} \qquad (2)$$

This must be equated to the heat flux through the skin/fat layer and into the environment. In terms of the heat transfer rate per unit length of forearm, q_i', and the total resistance for a unit length, R'_{tot},

$$q_i'' = \frac{q_i'}{2\pi r_1} = \frac{1}{2\pi r_1}\frac{T_i - T_\infty}{R'_{tot}} \qquad (3)$$

As in Example 3.1 and for exposure of the skin to the air, R'_{tot} accounts for conduction through the skin/fat layer in series with heat transfer by convection and radiation, which act in parallel with each other. Here the conduction resistance is for a radial geometry. Thus, it is

$$R'_{tot} = \frac{\ln(r_2/r_1)}{2\pi k_{sf}} + \frac{1}{2\pi r_2}\left(\frac{1}{1/h} + \frac{1}{1/h_r}\right)^{-1} = \frac{\ln(r_2/r_1)}{2\pi k_{sf}} + \frac{1}{2\pi r_2}\left(\frac{1}{h + h_r}\right)$$

Using the values from Example 1.7 for air,

$$R'_{tot} = \frac{\ln(0.053\text{ m}/0.05\text{ m})}{2\times\pi\times 0.3\text{ W/m}\cdot\text{K}} + \frac{1}{2\times\pi\times 0.053\text{ m}}\left(\frac{1}{(2+5.9)\text{ W/m}^2\cdot\text{K}}\right) = 0.41\text{ m}\cdot\text{K/W}$$

Combining Equations 2 and 3 yields

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PROBLEM 3.167 (Cont.)

$$-k_m \theta_i \tilde{m} \frac{I_1(\tilde{m}r_1)}{I_0(\tilde{m}r_1)} = \frac{1}{2\pi r_1}\frac{T_i - T_\infty}{R'_{tot}}$$

This expression can be solved for T_i, recalling that T_i also appears in θ_i.

$$T_i = \frac{T_\infty I_0(\tilde{m}r_1) + k_m 2\pi r_1 \tilde{m} R'_{tot}\left(T_a + \frac{\dot{q}_m}{\omega\rho_b c_b}\right) I_1(\tilde{m}r_1)}{I_0(\tilde{m}r_1) + k_m 2\pi r_1 \tilde{m} R'_{tot} I_1(\tilde{m}r_1)}$$

where

$$\tilde{m} = \sqrt{\omega\rho_b c_b/k_m} = \left[0.0005\ \mathrm{s^{-1}} \times 1000\ \mathrm{kg/m^3} \times 3600\ \mathrm{J/kg\cdot K}\big/0.5\ \mathrm{W/m\cdot K}\right]^{1/2} = 60\ \mathrm{m^{-1}}$$

$$\frac{\dot{q}_m}{\omega\rho_b c_b} = \frac{700\ \mathrm{W/m^3}}{0.0005\ \mathrm{s^{-1}} \times 1000\ \mathrm{kg/m^3} \times 3600\ \mathrm{J/kg\cdot K}} = 0.389\ \mathrm{K}$$

and from Table B.5

$$I_0(\tilde{m}r_1) = I_0(60\ \mathrm{m^{-1}} \times 0.05\ \mathrm{m}) = I_0(3) = 4.88\,; \quad I_1(\tilde{m}r_1) = I_1(60\ \mathrm{m^{-1}} \times 0.05\ \mathrm{m}) = I_1(3) = 3.95$$

Thus,

$$T_i = \left[\left[\begin{array}{l} 24^\circ\mathrm{C} \times 4.88 + 0.5\ \mathrm{W/m\cdot K} \\ \times 2 \times \pi \times 0.05\ \mathrm{m} \times 60\ \mathrm{m^{-1}} \\ \times 0.41\ \mathrm{m\cdot K/W}\left(37 + 0.389\right)^\circ\mathrm{C} \times 3.95 \end{array}\right]\Big/\left[\begin{array}{l} 4.88 + 0.5\ \mathrm{W/m\cdot K} \\ \times 2 \times \pi \times 0.05\ \mathrm{m} \times 60\ \mathrm{m^{-1}} \\ \times 0.41\ \mathrm{m\cdot K/W} \times 3.95 \end{array}\right]\right] = 34.2^\circ\mathrm{C} \qquad <$$

(c) The maximum temperature occurs at the centerline of the forearm, r = 0, thus from Equation 1, with $I_0(0) = 1$,

$$T = T_a + \frac{\dot{q}_m}{\omega\rho_b c_b} + \left(T_i - T_a - \frac{\dot{q}_m}{\omega\rho_b c_b}\right)\frac{1}{I_0(\tilde{m}r_1)} \qquad <$$

$$= 37^\circ\mathrm{C} + 0.389^\circ\mathrm{C} + \left(34.2^\circ\mathrm{C} - 37^\circ\mathrm{C} - 0.389^\circ\mathrm{C}\right) \times \frac{1}{4.88} = 36.7^\circ\mathrm{C}$$

COMMENTS: (1) The maximum temperature is very close to the core body temperature of 37ºC, as would be expected. (2) Pennes [17] conducted an experimental investigation of the temperature distribution in human forearms, by inserting thermocouples into living subjects.