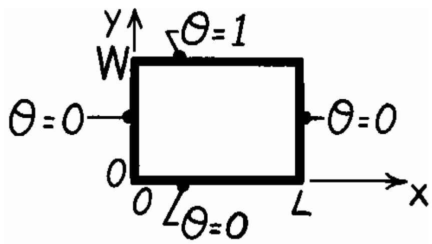
PROBLEM 4.1

KNOWN: Method of separation of variables for two-dimensional, steady-state conduction.

FIND: Show that negative or zero values of λ^2, the separation constant, result in solutions which cannot satisfy the boundary conditions.

SCHEMATIC:



ASSUMPTIONS: (1) Two-dimensional, steady-state conduction, (2) Constant properties.

ANALYSIS: From Section 4.2, identification of the separation constant λ^2 leads to the two ordinary differential equations, 4.6 and 4.7, having the forms

$$\frac{d^2X}{dx^2} + \lambda^2 X = 0 \qquad \frac{d^2Y}{dy^2} - \lambda^2 Y = 0 \tag{1,2}$$

and the temperature distribution is

$$\theta(x,y) = X(x) \cdot Y(y). \tag{3}$$

Consider now the situation when $\lambda^2 = 0$. From Eqs. (1), (2), and (3), find that

$$X = C_1 + C_2 x, \quad Y = C_3 + C_4 y \quad \text{and} \quad \theta(x,y) = (C_1 + C_2 x)\ (C_3 + C_4 y). \tag{4}$$

Evaluate the constants - C_1, C_2, C_3 and C_4 - by substitution of the boundary conditions:

$$\begin{array}{lll} x = 0: & \theta(0,y) = (C_1 + C_2 \cdot 0)(C_3 + C_4 \cdot y) = 0 & C_1 = 0 \\ y = 0: & \theta(x,0) = (0 + C_2 \cdot x)(C_3 + C_4 \cdot 0) = 0 & C_3 = 0 \\ x = L: & \theta(L,0) = (0 + C_2 \cdot L)(0 + C_4 \cdot y) = 0 & C_2 = 0 \\ y = W: & \theta(x,W) = (0 + 0 \cdot x)(0 + C_4 \cdot W) = 1 & 0 \neq 1 \end{array}$$

The last boundary condition leads to an impossibility ($0 \neq 1$). We therefore conclude that a λ^2 value of zero will not result in a form of the temperature distribution which will satisfy the boundary conditions. Consider now the situation when $\lambda^2 < 0$. The solutions to Eqs. (1) and (2) will be

$$X = C_5 e^{-\lambda x} + C_6 e^{+\lambda x}, \qquad Y = C_7 \cos \lambda y + C_8 \sin \lambda y \tag{5,6}$$

and

$$\theta(x,y) = \left[C_5 e^{-\lambda x} + C_6 e^{+\lambda x}\right]\ \left[C_7 \cos \lambda y + C_8 \sin \lambda y\right]. \tag{7}$$

Evaluate the constants for the boundary conditions.

$$\begin{array}{lll} y = 0: & \theta(x,0) = \left[C_5 e^{-\lambda x} + C_6 e^{-\lambda x}\right]\ \left[C_7 \cos 0 + C_8 \sin 0\right] = 0 & C_7 = 0 \\ x = 0: & \theta(0,y) = \left[C_5 e^{0} + C_6 e^{0}\right]\ \left[0 + C_8 \sin \lambda y\right] = 0 & C_8 = 0 \end{array}$$

If $C_8 = 0$, a trivial solution results or $C_5 = -C_6$.

$$x = L: \quad \theta(L,y) = C_5 \left[e^{-xL} - e^{+xL}\right] C_8 \sin \lambda y = 0.$$

From the last boundary condition, we require C_5 or C_8 is zero; either case leads to a trivial solution with either no x or y dependence.