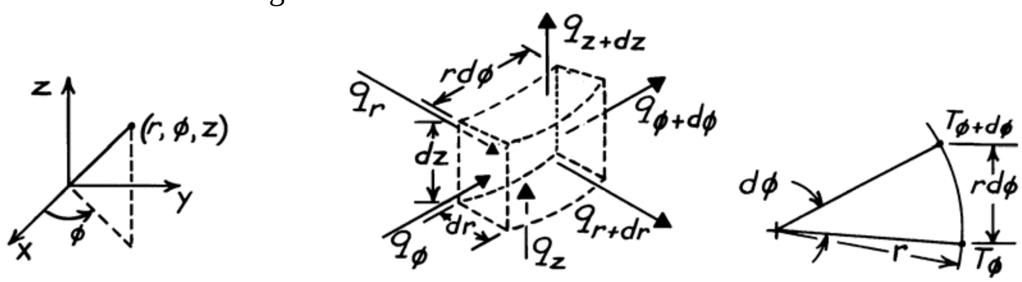
PROBLEM 2.35

KNOWN: Three-dimensional system – described by cylindrical coordinates (r,ϕ,z) – experiences transient conduction and internal heat generation.

FIND: Heat diffusion equation.

SCHEMATIC: See also Fig. 2.12.



ASSUMPTIONS: (1) Homogeneous medium.

ANALYSIS: Consider the differential control volume identified above having a volume given as V = dr·rdϕ·dz. From the conservation of energy requirement,

$$q_r - q_{r+dr} + q_\phi - q_{\phi+d\phi} + q_z - q_{z+dz} + \dot{E}_g = \dot{E}_{st}. \qquad (1)$$

The generation and storage terms, both representing volumetric phenomena, are

$$\dot{E}_g = \dot{q}V = \dot{q}\left(dr \cdot rd\phi \cdot dz\right) \qquad \dot{E}_g = \rho Vc\partial T/\partial t = \rho\left(dr \cdot rd\phi \cdot dz\right)c\,\partial T/\partial t. \qquad (2,3)$$

Using a Taylor series expansion, we can write

$$q_{r+dr} = q_r + \frac{\partial}{\partial r}\left(q_r\right)dr, \quad q_{\phi+d\phi} = q_\phi + \frac{\partial}{\partial \phi}\left(q_\phi\right)d\phi, \quad q_{z+dz} = q_z + \frac{\partial}{\partial z}\left(q_z\right)dz. \qquad (4,5,6)$$

Using Fourier's law, the expressions for the conduction heat rates are

$$q_r = -kA_r\partial T/\partial r = -k\left(rd\phi \cdot dz\right)\partial T/\partial r \qquad (7)$$

$$q_\phi = -kA_\phi\partial T/r\partial\phi = -k\left(dr \cdot dz\right)\partial T/r\partial\phi \qquad (8)$$

$$q_z = -kA_z\partial T/\partial z = -k\left(dr \cdot rd\phi\right)\partial T/\partial z. \qquad (9)$$

Note from the above, right schematic that the gradient in the ϕ-direction is ∂T/r∂ϕ and not ∂T/∂ϕ. Substituting Eqs. (2), (3) and (4), (5), (6) into Eq. (1),

$$-\frac{\partial}{\partial r}\left(q_r\right)dr - \frac{\partial}{\partial\phi}\left(q_\phi\right)d\phi - \frac{\partial}{\partial z}\left(q_z\right)dz + \dot{q}\;dr \cdot rd\phi \cdot dz = \rho\left(dr \cdot rd\phi \cdot dz\right)c\frac{\partial T}{\partial t}. \qquad (10)$$

Substituting Eqs. (7), (8) and (9) for the conduction rates, find

$$-\frac{\partial}{\partial r}\left[-k\left(rd\phi \cdot dz\right)\frac{\partial T}{\partial r}\right]dr - \frac{\partial}{\partial \phi}\left[-k\left(drdz\right)\frac{\partial T}{r\partial\phi}\right]d\phi - \frac{\partial}{\partial z}\left[-k\left(dr \cdot rd\phi\right)\frac{\partial T}{\partial z}\right]dz$$

$$+\dot{q}\;dr \cdot rd\phi \cdot dz = \rho\left(dr \cdot rd\phi \cdot dz\right)c\frac{\partial T}{\partial t}. \qquad (11)$$

Dividing Eq. (11) by the volume of the CV, Eq. 2.26 is obtained.

$$\frac{1}{r}\frac{\partial}{\partial r}\left[kr\frac{\partial T}{\partial r}\right] + \frac{1}{r^2}\frac{\partial}{\partial\phi}\left[k\frac{\partial T}{\partial\phi}\right] + \frac{\partial}{\partial z}\left[k\frac{\partial T}{\partial z}\right] + \dot{q} = \rho c\frac{\partial T}{\partial t}$$

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