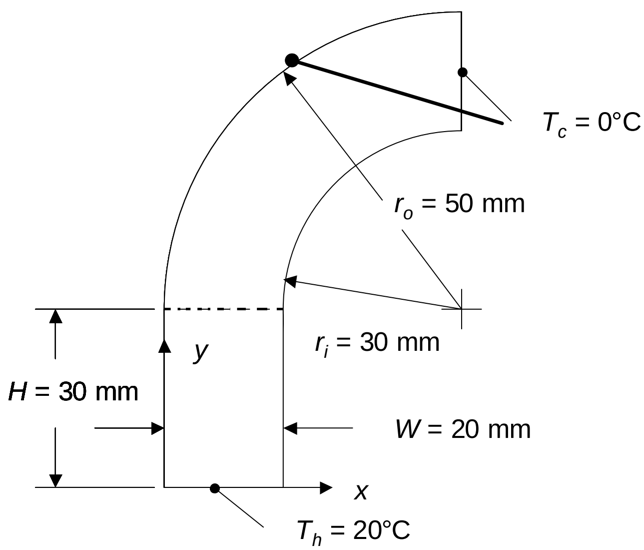
PROBLEM 4.59

KNOWN: Dimensions of a two-dimensional object with isothermal and adiabatic boundaries.

FIND: Conduction heat transfer rate per unit depth from the hot surface to the cold surface.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) Constant properties, (3) No internal generation, (4) Two-dimensional conduction.

ANALYSIS: We may combine heat fluxes determined from Fourier's law with expressions for the size of the control surfaces of the various control volumes to determine the heat rate per unit depth into each control volume in the discretized domain, which is shown on the next page. Application of conservation of energy for each control volume yields the expression $\dot{E}_{\text{in}} = 0$. Note that energy balances for nodes 10, 11, and 12 are included in both the rectangular and cylindrical sub-domains. These energy balances couple the solutions together.

Rectangular Sub-Domain. For the rectangular sub-domain, application of Fourier's law in Cartesian coordinates along with conservation of energy yields the following finite difference equations.

Nodes 1, 2 and 3: $T_1 = T_2 = T_3 = T_h = 20°C$.

Node 4: $$k\frac{(T_1 - T_4)}{\Delta y}\frac{\Delta x}{2} + k\frac{(T_5 - T_4)}{\Delta x}\Delta y + k\frac{(T_7 - T_4)}{\Delta y}\frac{\Delta x}{2} = 0$$

Node 5: $$k\frac{(T_2 - T_5)}{\Delta y}\Delta x + k\frac{(T_6 - T_5)}{\Delta x}\Delta y + k\frac{(T_8 - T_5)}{\Delta y}\Delta x + k\frac{(T_4 - T_5)}{\Delta x}\Delta y = 0$$

Node 6: $$k\frac{(T_3 - T_6)}{\Delta y}\frac{\Delta x}{2} + k\frac{(T_5 - T_6)}{\Delta x}\Delta y + k\frac{(T_9 - T_6)}{\Delta y}\frac{\Delta x}{2} = 0$$

Node 7: $$k\frac{(T_4 - T_7)}{\Delta y}\frac{\Delta x}{2} + k\frac{(T_8 - T_7)}{\Delta x}\Delta y + k\frac{(T_{10} - T_7)}{\Delta y}\frac{\Delta x}{2} = 0$$

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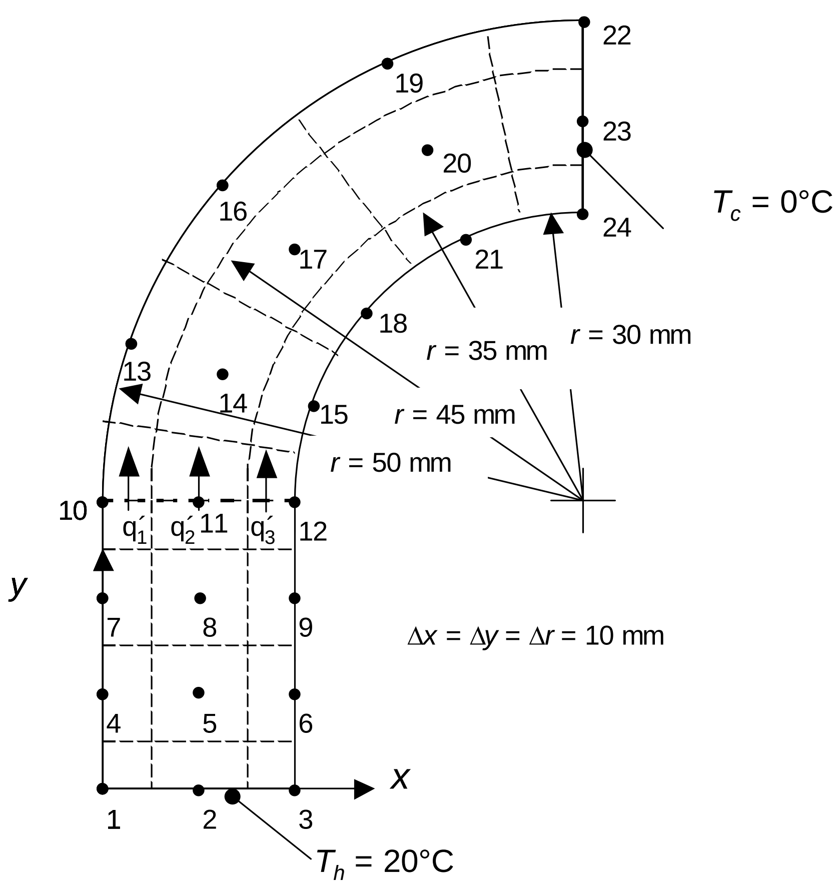
Node 8: $$k\frac{(T_5-T_8)}{\Delta y}\frac{\Delta x}{2}+k\frac{(T_9-T_8)}{\Delta x}\Delta y+k\frac{(T_{11}-T_8)}{\Delta y}\Delta x+k\frac{(T_7-T_8)}{\Delta x}\Delta y=0$$

Node 9: $$k\frac{(T_6-T_9)}{\Delta y}\frac{\Delta x}{2}+k\frac{(T_8-T_9)}{\Delta x}\Delta y+k\frac{(T_{12}-T_9)}{\Delta y}\frac{\Delta x}{2}=0$$

Node 10: $$k\frac{(T_7-T_{10})}{\Delta y}\frac{\Delta x}{2}+k\frac{(T_{11}-T_{10})}{\Delta x}\frac{\Delta y}{2}-q_1^{'}=0$$

Node 11: $$k\frac{(T_8-T_{11})}{\Delta y}\Delta x+k\frac{(T_{10}-T_{11})}{\Delta x}\frac{\Delta y}{2}+k\frac{(T_{12}-T_{11})}{\Delta x}\frac{\Delta y}{2}-q_2^{'}=0$$

Node 12: $$k\frac{(T_9-T_{12})}{\Delta y}\frac{\Delta x}{2}+k\frac{(T_{11}-T_{12})}{\Delta x}\frac{\Delta y}{2}-q_3^{'}=0$$



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Cylindrical Sub-Domain. We begin by recalling that Fourier's law for the cylindrical coordinate system yields

$$q_r'' = -k\frac{\partial T}{\partial r} \approx -k\frac{\Delta T}{\Delta r}; \quad q_\phi'' = -\frac{k}{r}\frac{\partial T}{\partial \phi} \approx -\frac{k}{r}\frac{\Delta T}{\Delta \phi}$$

and the areas through which conduction occurs in the radial direction increase as the radius increases.

Nodes 10, 13, 16, 19, 22, 23 and 24: $T_{10} = T_{13} = T_{16} = T_{19} = T_{22} = T_{23} = T_{24} = T_c = 0°C$

Node 11: $q_2' + k\frac{(T_{12}-T_{11})}{\Delta r}\cdot 3.5\Delta r\cdot\frac{\Delta\phi}{2} + k\frac{(T_{14}-T_{11})}{4\Delta r\Delta\phi}\cdot\Delta r + k\frac{(T_{10}-T_{11})}{\Delta r}\cdot 4.5\Delta r\cdot\frac{\Delta\phi}{2} = 0$

Node 12: $q_3' + k\frac{(T_{11}-T_{12})}{\Delta r}\cdot 3.5\Delta r\cdot\frac{\Delta\phi}{2} + k\frac{(T_{15}-T_{12})}{3\Delta r\Delta\phi}\cdot\frac{\Delta r}{2} = 0$

Node 14: $k\frac{(T_{11}-T_{14})}{4\Delta r\Delta\phi}\cdot\Delta r + k\frac{(T_{15}-T_{14})}{\Delta r}\cdot 3.5\Delta r\Delta\phi + k\frac{(T_{17}-T_{14})}{4\Delta r\Delta\phi}\Delta r + k\frac{(T_{13}-T_{14})}{\Delta r}\cdot 4.5\Delta r\Delta\phi = 0$

Node 15: $k\frac{(T_{12}-T_{15})}{3\Delta r\Delta\phi}\frac{\Delta r}{2} + k\frac{(T_{14}-T_{15})}{\Delta r}\cdot 3.5\Delta r\Delta\phi + k\frac{(T_{18}-T_{15})}{3\Delta r\Delta\phi}\frac{\Delta r}{2} = 0$

Node 17: $k\frac{(T_{14}-T_{17})}{4\Delta r\Delta\phi}\cdot\Delta r + k\frac{(T_{18}-T_{17})}{\Delta r}\cdot 3.5\Delta r\Delta\phi + k\frac{(T_{20}-T_{17})}{4\Delta r\Delta\phi}\Delta r + k\frac{(T_{16}-T_{17})}{\Delta r}\cdot 4.5\Delta r\Delta\phi = 0$

Node 18: $k\frac{(T_{15}-T_{18})}{3\Delta r\Delta\phi}\frac{\Delta r}{2} + k\frac{(T_{17}-T_{18})}{\Delta r}\cdot 3.5\Delta r\Delta\phi + k\frac{(T_{21}-T_{18})}{3\Delta r\Delta\phi}\frac{\Delta r}{2} = 0$

Node 20: $k\frac{(T_{17}-T_{20})}{4\Delta r\Delta\phi}\Delta r + k\frac{(T_{21}-T_{20})}{\Delta r}\cdot 3.5\Delta r\Delta\phi + k\frac{(T_{23}-T_{20})}{4\Delta r\Delta\phi}\Delta r = 0$

Node 21: $k\frac{(T_{18}-T_{21})}{3\Delta r\Delta\phi}\frac{\Delta r}{2} + k\frac{(T_{20}-T_{21})}{\Delta r}\cdot 3.5\Delta r\Delta\phi + k\frac{(T_{24}-T_{21})}{3\Delta r\Delta\phi}\frac{\Delta r}{2} = 0$

Note that energy balances for nodes 10, 11 and 12 are included in both the rectangular and cylindrical sub-domains. These energy balances couple the solutions for the two sub-domains together.

The preceding finite difference equations may be solved simultaneously with the *IHT* code provided in the Comment yielding the following temperatures and $q' = 114.5$ W/m. <

The nodal temperatures are:

T_1 = 20.00°C	T_2 = 20.00°C	T_3 = 20.00°C	T_4 = 14.11°C	T_5 = 14.29°C	T_6 = 14.41°C
T_7 = 7.86°C	T_8 = 8.66°C	T_9 = 9.03°C	T_{10} = 0.00°C	T_{11} = 3.46°C	T_{12} = 4.40°C
T_{13} = 0.00°C	T_{14} = 1.04°C	T_{15} = 1.59°C	T_{16} = 0.00°C	T_{17} = 0.33°C	T_{18} = 0.54°C
T_{19} = 0.00°C	T_{20} = 0.10°C	T_{21} = 0.16°C	T_{22} = 0.00°C	T_{23} = 0.00°C	T_{24} = 0.00°C

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COMMENTS: (1) The *IHT* code is listed below. For each control volume, we note that $\dot{E}_{in} = 0$ and $\Delta y = \Delta x$, yielding the following energy balances for all but the isothermal nodes.

```
// Input Parameters

Th = 20
Tc = 0
k = 10
deltaphi = pi/8

// Node Equations for Rectangle

//Node 1
T1 = Th
//Node 2
T2 = Th
//Node 3
T3 = Th
//Node 4
(T1 - T4)/2 + (T5 - T4) + (T7 - T4)/2 = 0
//Node 5
(T2 - T5) + (T6 - T5) + (T8 - T5) + (T4 - T5) = 0
//Node 6
(T3 - T6)/2 + (T5 - T6) + (T9 - T6)/2 = 0
//Node 7
(T4 - T7)/2 + (T8 - T7) + (T10 - T7)/2 = 0
//Node 8
(T5 - T8) + (T9 - T8) + (T11 - T8) + (T7 - T8) = 0
//Node 9
(T6 - T9)/2 + (T8 - T9) + (T12 - T9)/2 = 0

//Nodes Common to Both Sub-Domains (Rectangle)
//Node 10
(T7 - T10)/2 + (T11 - T10)/2 -qprime1/k = 0
//Node 11
(T8 - T11)+(T10 - T11)/2 + (T12 - T11)/2 - qprime2/k = 0
//Node 12
(T9 - T12)/2 + (T11 - T12)/2 - qprime3/k = 0

//Nodes Common to Both Sub-Domains (Cylindrical)
//Node 10
T10 = Tc
//Node 11
qprime2/k + (T12 - T11)*3.5*deltaphi/2 + (T14 - T11)/4/deltaphi + (T10 - T11)*4.5*deltaphi/2 = 0
//Node 12
qprime3/k + (T11 - T12)*3.5*deltaphi/2 + (T15 - T12)/3/deltaphi/2 = 0


// Node Equations for Cylindrical Region

//Node 13
T13 = Tc
//Node 14
(T11 - T14)/4/deltaphi + (T15 - T14)*3.5*deltaphi + (T17 - T14)/4/deltaphi + (T13 - T14)*4.5*deltaphi =
0
//Node 15
(T12 - T15)/3/deltaphi/2 + (T14 - T15)*3.5*deltaphi + (T18 - T15)/3/deltaphi/2 = 0
//Node 16
T16 = Tc
//Node 17
(T14 - T17)/4/deltaphi + (T18 - T17)*3.5*deltaphi + (T20 - T17)/4/deltaphi + (T16 - T17)*4.5*deltaphi =
0
//Node 18
(T15 - T18)/3/deltaphi/2 + (T17 - T18)*3.5*deltaphi + (T21 - T18)/3/deltaphi/2 = 0
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```
//Node 19
T19 = Tc
//Node 20
(T17 - T20)/4/deltaphi + (T21 - T20)*3.5*deltaphi + (T23 - T20)/4/deltaphi + (T19 - T20)*4.5*deltaphi =
0
//Node 21
(T18 - T21)/3/deltaphi/2 + (T20 - T21)*3.5*deltaphi + (T24 - T21)/3/deltaphi/2 = 0
//Node 22
T22 = Tc
//Node 23
T23 = Tc
//Node 24
T24 = Tc

qprime = qprime1 + qprime2 + qprime3
```

(2) The shape factor for the geometry is $S = \dfrac{q'}{k(Th - Tc)} = \dfrac{114.5\ \text{W/m}}{10\text{W/m}\cdot\text{K} \times 20\text{K}} = 0.573$