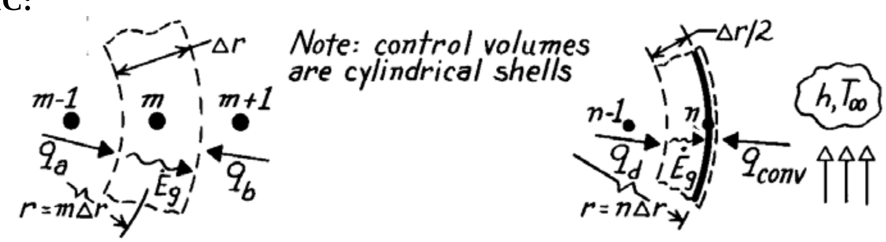
PROBLEM 4.43

KNOWN: Conduction in a one-dimensional (radial) *cylindrical* coordinate system with volumetric generation.

FIND: Finite-difference equation for (a) Interior node, m, and (b) Surface node, n, with convection.

SCHEMATIC:



(a) Interior node, m   (b) Surface node with convection, n

ASSUMPTIONS: (1) Steady-state, one-dimensional (radial) conduction in *cylindrical* coordinates, (2) Constant properties.

ANALYSIS: (a) The network has nodes spaced at equal Δr increments with m = 0 at the center; hence, r = mΔr (or nΔr). The control volume is $V = 2\pi\ r \cdot \Delta r \cdot \ell = 2\pi(m\Delta r)\Delta r \cdot \ell$. The energy balance is $\dot{E}_{in} + \dot{E}_g = q_a + q_b + \dot{q}V = 0$

$$k\left[2\pi\left[r - \frac{\Delta r}{2}\right]\ell\right]\frac{T_{m-1} - T_m}{\Delta r} + k\left[2\pi\left[r + \frac{\Delta r}{2}\right]\ell\right]\frac{T_{m+1} - T_m}{\Delta r} + \dot{q}\left[2\pi(m\Delta r)\Delta r\ell\right] = 0.$$

Recognizing that r = mΔr, canceling like terms, and regrouping find

$$\left[m - \frac{1}{2}\right]T_{m-1} + \left[m + \frac{1}{2}\right]T_{m+1} - 2mT_m + \frac{\dot{q}m\Delta r^2}{k} = 0. \qquad <$$

(b) The control volume for the surface node is $V = 2\pi\ r \cdot (\Delta r/2) \cdot \ell$. The energy balance is $\dot{E}_{in} + \dot{E}_g = q_d + q_{conv} + \dot{q}V = 0$. Use Fourier's law to express q_d and Newton's law of cooling for q_{conv} to obtain

$$k\left[2\pi\left[r - \frac{\Delta r}{2}\right]\ell\right]\frac{T_{n-1} - T_n}{\Delta r} + h\left[2\pi\ r\ell\right]\left(T_\infty - T_n\right) + \dot{q}\left[2\pi(n\Delta r)\frac{\Delta r}{2}\ell\right] = 0.$$

Let r = nΔr, cancel like terms and regroup to find

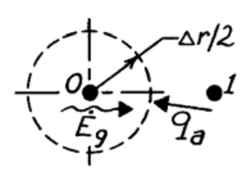
$$\left[n - \frac{1}{2}\right]T_{n-1} - \left[\left[n - \frac{1}{2}\right] + \frac{hn\Delta r}{k}\right]T_n + \frac{\dot{q}n\Delta r^2}{2k} + \frac{hn\Delta r}{k}T_\infty = 0. \qquad <$$

COMMENTS: (1) Note that when m or n becomes very large compared to ½, the finite-difference equation becomes independent of m or n. Then the cylindrical system approximates a rectangular one.

(2) The finite-difference equation for the center node (m = 0) needs to be treated as a special case. The control volume is

$V = \pi(\Delta r/2)^2\ \ell$ and the energy balance is

$$\dot{E}_{in} + \dot{E}_g = q_a + \dot{q}V = k\left[2\pi\left[\frac{\Delta r}{2}\right]\ell\right]\frac{T_1 - T_0}{\Delta r} + \dot{q}\left[\pi\left[\frac{\Delta r}{2}\right]^2\ell\right] = 0.$$



Regrouping, the finite-difference equation is $-T_o + T_1 + \frac{\dot{q}\Delta r^2}{4k} = 0.$