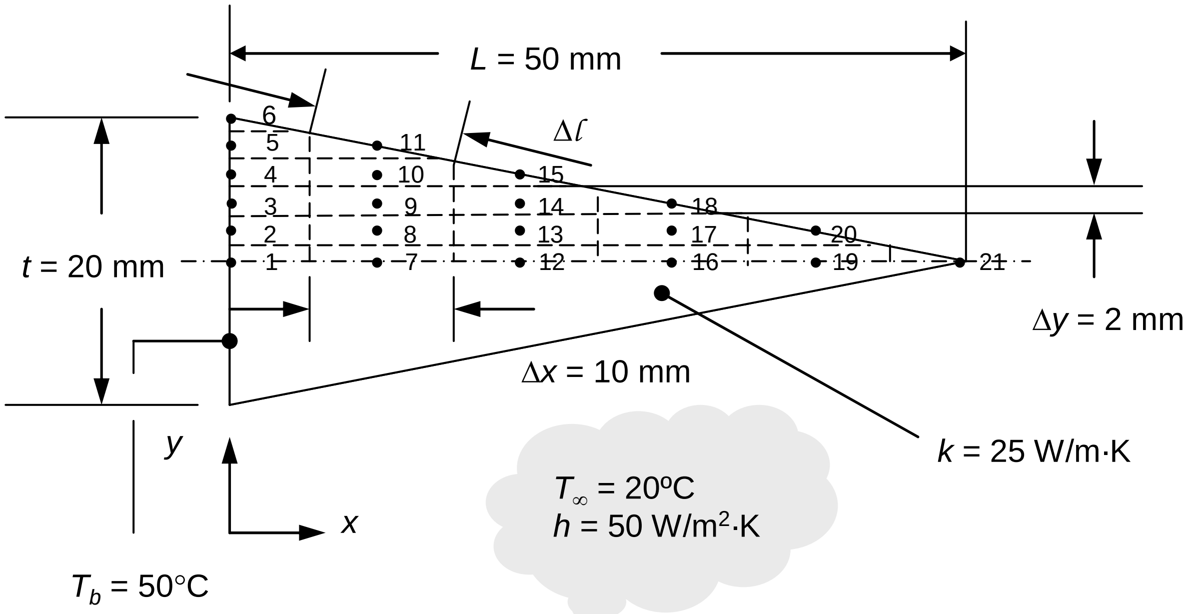
PROBLEM 4.66

KNOWN: Dimensions of a two-dimensional, straight triangular fin. Fin base and ambient temperatures, thermal conductivity and heat transfer coefficient.

FIND: Fin efficiency by using a finite difference solution with specified grid.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) Constant properties, (3) No internal generation, (4) Two-dimensional conduction.

ANALYSIS: We may combine heat fluxes determined from Fourier's law and Newton's law of cooling with expressions for the size of the control surfaces of the various control volumes to determine the heat rate per unit depth into each control volume within the discretized domain. Application of conservation of energy for each control volume yields the expression $\dot{E}_{\text{in}} = 0$. Note that $\Delta l = \sqrt{\Delta x^2 + \Delta y^2}$ = 10.198 mm.

Nodes 1, 2, 3, 4, 5 and 6: $T_1 = T_2 = T_3 = T_4 = T_5 = T_6 = T_b = 50°C$.

Node 7: $$\frac{k(T_1 - T_7)(\Delta y/2)}{\Delta x} + \frac{k(T_8 - T_7)\Delta x}{\Delta y} + \frac{k(T_{12} - T_7)(\Delta y/2)}{\Delta x} = 0$$

Node 8: $$\frac{k(T_2 - T_8)(\Delta y)}{\Delta x} + \frac{k(T_7 - T_8)\Delta x}{\Delta y} + \frac{k(T_{13} - T_8)(\Delta y)}{\Delta x} + \frac{k(T_9 - T_8)(\Delta x)}{\Delta y} = 0$$

Node 9: $$\frac{k(T_3 - T_9)(\Delta y)}{\Delta x} + \frac{k(T_8 - T_9)\Delta x}{\Delta y} + \frac{k(T_{14} - T_9)(\Delta y)}{\Delta x} + \frac{k(T_{10} - T_9)(\Delta x)}{\Delta y} = 0$$

Node 10: $$\frac{k(T_4 - T_{10})(\Delta y)}{\Delta x} + \frac{k(T_9 - T_{10})\Delta x}{\Delta y} + \frac{k(T_{15} - T_{10})(\Delta y)}{\Delta x} + \frac{k(T_{11} - T_{10})(\Delta x)}{\Delta y} = 0$$

Node 11: $$\frac{k(T_5 - T_{11})(\Delta y)}{\Delta x} + \frac{k(T_{10} - T_{11})\Delta x}{\Delta y} + h\Delta l(T_\infty - T_{11}) = 0$$

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Node 12: $$\frac{k(T_7-T_{12})(\Delta y/2)}{\Delta x}+\frac{k(T_{13}-T_{12})\Delta x}{\Delta y}+\frac{k(T_{16}-T_{12})(\Delta y/2)}{\Delta x}=0$$

Node 13: $$\frac{k(T_8-T_{13})(\Delta y)}{\Delta x}+\frac{k(T_{12}-T_{13})\Delta x}{\Delta y}+\frac{k(T_{17}-T_{13})(\Delta y)}{\Delta x}+\frac{k(T_{14}-T_{13})(\Delta x)}{\Delta y}=0$$

Node 14: $$\frac{k(T_9-T_{14})(\Delta y)}{\Delta x}+\frac{k(T_{13}-T_{14})\Delta x}{\Delta y}+\frac{k(T_{18}-T_{14})(\Delta y)}{\Delta x}+\frac{k(T_{15}-T_{14})(\Delta x)}{\Delta y}=0$$

Node 15: $$\frac{k(T_{10}-T_{15})(\Delta y)}{\Delta x}+\frac{k(T_{14}-T_{15})\Delta x}{\Delta y}+h\Delta l(T_\infty-T_{15})=0$$

Node 16: $$\frac{k(T_{12}-T_{16})(\Delta y/2)}{\Delta x}+\frac{k(T_{17}-T_{16})\Delta x}{\Delta y}+\frac{k(T_{19}-T_{16})(\Delta y/2)}{\Delta x}=0$$

Node 17: $$\frac{k(T_{13}-T_{17})(\Delta y)}{\Delta x}+\frac{k(T_{16}-T_{17})\Delta x}{\Delta y}+\frac{k(T_{20}-T_{17})(\Delta y)}{\Delta x}+\frac{k(T_{18}-T_{17})(\Delta x)}{\Delta y}=0$$

Node 18: $$\frac{k(T_{14}-T_{18})(\Delta y)}{\Delta x}+\frac{k(T_{17}-T_{18})\Delta x}{\Delta y}+h\Delta l(T_\infty-T_{18})=0$$

Node 19: $$\frac{k(T_{16}-T_{19})(\Delta y/2)}{\Delta x}+\frac{k(T_{20}-T_{19})\Delta x}{\Delta y}+\frac{k(T_{21}-T_{19})(\Delta y/2)}{\Delta x}=0$$

Node 20: $$\frac{k(T_{17}-T_{20})(\Delta y)}{\Delta x}+\frac{k(T_{19}-T_{20})\Delta x}{\Delta y}+h\Delta l(T_\infty-T_{20})=0$$

Node 21: $$\frac{k(T_{19}-T_{21})(\Delta y/2)}{\Delta x}+h\left(\Delta l/2\right)(T_\infty-T_{21})=0$$

The fin heat rate per unit length is evaluated by considering conduction into its base expressed as

$$q_f^{'}=2\left(k\frac{(T_1-T_7)(\Delta y/2)}{\Delta x}+k\frac{(T_2-T_8)(\Delta y)}{\Delta x}+k\frac{(T_3-T_9)(\Delta y)}{\Delta x}+k\frac{(T_4-T_{10})(\Delta y)}{\Delta x}+k\frac{(T_5-T_{11})(\Delta y)}{\Delta x}\right)$$

where the factor of two is due to heat transfer in the bottom half of the fin. The result is $q_f^{'}=109$ W/m . The fin efficiency is evaluated using Eq. 3.91 yielding

$$n_f=\frac{q_f^{'}}{hA_f\theta_b}=\frac{q_f^{'}}{h\left(2\sqrt{L^2+(t/2)^2}\right)\theta_b}=\frac{109\text{ W/m}}{50\text{W/m}^2\cdot\text{K}\left(2\sqrt{(50\times10^{-3}\text{m})^2+(20\times10^{-3}\text{m}/2)^2}\right)30\text{ K}}=0.71 \quad <$$

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From Fig. 3.19 we find $L_c = 50 \times 10^{-3}$ m, $A_p = Lt/2 = (50 \times 10^{-3}\text{ m} \times 20 \times 10^{-3}\text{ m})/2 = 500 \times 10^{-6}\text{ m}^2$. Therefore,

$$L_c^{3/2}\left(\frac{h}{kA_p}\right)^{1/2} = (50\times10^{-3}\text{m})^{3/2} \times \left(\frac{50\text{ W/m}^2\cdot\text{K}}{25\text{ W/m}\cdot\text{K}\times500\times10^{-6}\text{m}^2}\right)^{1/2} = 0.707$$

and $n_f \approx 0.78$. The comparison between the calculated value and the value from the figure is reasonable. The difference may be attributed to convective loss from the fin adjacent to the base that is not accounted for in the finite difference solution, as discussed in Comment 3 below, or more generally, to the relatively coarse nodal mesh.

COMMENTS: (1) The nodal temperatures are:

T_6 = 50°C					
T_5 = 50°C	T_{11} = 47.57°C				
T_4 = 50°C	T_{10} = 47.58°C	T_{15} = 45.28°C			
T_3 = 50°C	T_9 = 47.59°C	T_{14} = 45.29°C	T_{18} = 43.10°C		
T_2 = 50°C	T_8 = 47.60°C	T_{13} = 45.30°C	T_{17} = 43.11°C	T_{20} = 41.03°C	
T_1 = 50°C	T_7 = 47.60°C	T_{12} = 45.30°C	T_{16} = 43.11°C	T_{19} = 41.03°C	T_{21} = 39.09°C

Note the nearly uniform cross-sectional temperatures within the fin. Temperatures near the centerline are only slightly warmer than corresponding temperatures at a particular *x*-location nearer to the convectively-cooled fin surface. (2) The *IHT* code is listed below.

```
// Input Parameters
k = 25
Tinf = 20
Tbase = 50
delx = 0.01
dely = 0.002
h = 50
dell = sqrt(delx^2 + dely^2)

//Nodal Energy Balance Equations

//Node 1
T1 = Tbase
//Node 2
T2 = Tbase
//Node 3
T3 = Tbase
//Node 4
T4 = Tbase
//Node 5
T5 = Tbase
//Node 6
T6 = Tbase
//Node 7
k*(T1 - T7)*dely/2/delx + k*(T8 - T7)*delx/dely + k*(T12 - T7)*dely/2/delx = 0
//Node 8
k*(T2 - T8)*dely/delx + k*(T7 - T8)*delx/dely + k*(T13 - T8)*dely/delx + k*(T9 - T8)*delx/dely = 0
//Node 9
k*(T3 - T9)*dely/delx + k*(T8 - T9)*delx/dely + k*(T14 - T9)*dely/delx + k*(T10 - T9)*delx/dely = 0
//Node 10
k*(T4 - T10)*dely/delx + k*(T9 - T10)*delx/dely + k*(T15 - T10)*dely/delx + k*(T11 - T10)*delx/dely = 0
//Node 11
k*(T5 - T11)*dely/delx + k*(T10 - T11)*delx/dely + h*dell*(Tinf - T11) = 0
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```
//Node 12
k*(T7 - T12)*dely/2/delx + k*(T13 - T12)*delx/dely + k*(T16 - T12)*dely/2/delx = 0
//Node 13
k*(T8 - T13)*dely/delx + k*(T12 - T13)*delx/dely + k*(T17 - T13)*dely/delx + k*(T14 - T13)*delx/dely = 0
//Node 14
k*(T9 - T14)*dely/delx + k*(T13 - T14)*delx/dely + k*(T18 - T14)*dely/delx + k*(T15 - T14)*delx/dely = 0
//Node 15
k*(T10 - T15)*dely/delx + k*(T14 - T15)*delx/dely + h*dell*(Tinf - T15) = 0
//Node 16
k*(T12 - T16)*dely/2/delx + k*(T17 - T16)*delx/dely + k*(T19 - T16)*dely/2/delx = 0
//Node 17
k*(T13 - T17)*dely/delx + k*(T16 - T17)*delx/dely + k*(T20 - T17)*dely/delx + k*(T18 - T17)*delx/dely = 0
//Node 18
k*(T14 - T18)*dely/delx + k*(T17 - T18)*delx/dely + h*dell*(Tinf - T18) = 0
//Node 19
k*(T16 - T19)*dely/2/delx + k*(T20 - T19)*delx/dely + k*(T21 - T19)*dely/2/delx = 0
//Node 20
k*(T17 - T20)*dely/delx + k*(T19 - T20)*delx/dely + h*dell*(Tinf - T20) = 0
//Node 21
k*(T19 - T21)*dely/2/delx + h*dell*(Tinf - T21)/2 = 0

//Fin Heat Transfer Rate

qfinhalf = k*(T1 - T7)*dely/2/delx + k*(T2 - T8)*dely/delx + k*(T3 - T9)*dely/delx + k*(T4 - T10)*dely/delx + k*(T5 -
T11)*dely/delx

qfin = 2*qfinhalf
```

(3) The finite difference equations do not account for convective losses from the fin in the exposed region of length $\Delta l / 2$ adjacent to the root of the fin. If this convective loss is estimated to be

$$q'_{est} = 2(\Delta l / 2) h (T_6 - T_\infty) = 2 \times (10.198 \times 10^{-3}\,\mathrm{m}) / 2 \times 50 \frac{\mathrm{W}}{\mathrm{m}^2 \cdot \mathrm{K}} \times (50°C - 20°C) = 15.3\ \mathrm{W/m}$$

The fin heat transfer rate increases to q'_f = 109 W/m + 15.3 W/m = 124.3 W/m and the fin efficiency increases to n_f = 0.81, slightly greater than the fin efficiency found from Fig. 3.19.