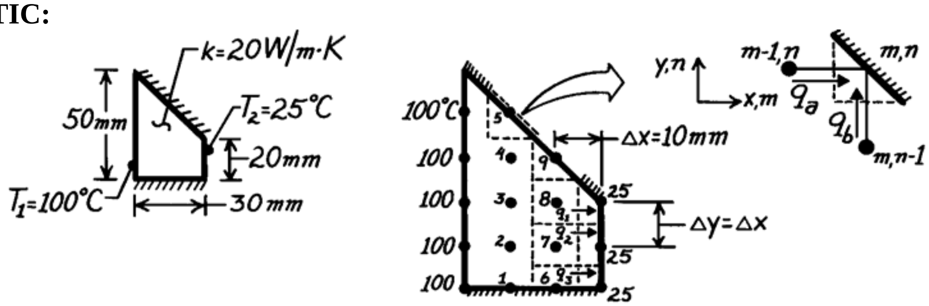
PROBLEM 4.85

KNOWN: Long bar with trapezoidal shape, uniform temperatures on two surfaces, and two insulated surfaces.

FIND: Heat transfer rate per unit length using finite-difference method with space increment of 10mm.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) Two-dimensional conduction, (3) Constant properties.

ANALYSIS: The heat rate can be found after the temperature distribution has been determined. Using the nodal network shown above with Δx = 10mm, nine finite-difference equations must be written. Nodes 1-4 and 6-8 are interior nodes and their finite-difference equations can be written directly from Eq. 4.29. For these nodes

$$T_{m,n+1} + T_{m,n-1} + T_{m+1,n} + T_{m-1,n} - 4T_{m,n} = 0 \qquad m = 1-4,\ 6-8. \tag{1}$$

For nodes 5 and 9 located on the diagonal, insulated boundary, the appropriate finite-difference equation follows from an energy balance on the control volume shown above (upper-right corner of schematic), $\dot{E}_{in} - \dot{E}_{out} = q_a + q_b = 0$

$$k(\Delta y \cdot 1)\frac{T_{m-1,n} - T_{m,n}}{\Delta x} + k(\Delta x \cdot 1)\frac{T_{m,n-1} - T_{m,n}}{\Delta y} = 0.$$

Since $\Delta x = \Delta y$, the finite-difference equation for nodes 5 and 9 is of the form

$$T_{m-1,n} + T_{m,n-1} - 2T_{m,n} = 0 \qquad m = 5,9. \tag{2}$$

The system of 9 finite-difference equations is first written in the form of Eqs. (1) or (2) and then written in explicit form for use with the Gauss-Seidel iteration method of solution; see Appendix D.

Node	*Finite-difference equation*	*Gauss-Seidel form*
1	$T_2+T_2+T_6+100-4T_1 = 0$	$T_1 = 0.5T_2+0.25T_6+25$
2	$T_3+T_1+T_7+100-4T_2 = 0$	$T_2 = 0.25(T_1+T_3+T_7)+25$
3	$T_4+T_2+T_8+100-4T_3 = 0$	$T_3 = 0.25(T_2+T_4+T_8)+25$
4	$T_5+T_3+T_9+100-4T_4 = 0$	$T_4 = 0.25(T_3+T_5+T_9)+25$
5	$100+T_4-2T_5 = 0$	$T_5 = 0.5T_4+50$
6	$T_7+T_7+25+T_1-4T_6 = 0$	$T_6 = 0.25T_1+0.5T_7+6.25$
7	$T_8+T_6+25+T_2-4T_7 = 0$	$T_7 = 0.25(T_2+T_6+T_8)+6.25$
8	$T_9+T_7+25+T_3-4T_8 = 0$	$T_8 = 0.25(T_3+T_7+T_9)+6.25$
9	$T_4+T_8-2T_9 = 0$	$T_9 = 0.5(T_4+T_8)$

Continued …

PROBLEM 4.85 (Cont.)

The iteration process begins after an initial guess (k = 0) is made. The calculations are shown in the table below.

k	T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8	T_9(°C)
0	75	75	80	85	90	50	50	60	75
1	75.0	76.3	80.0	86.3	92.5	50.0	52.5	57.5	72.5
2	75.7	76.9	80.0	86.3	93.2	51.3	52.2	57.5	71.9
3	76.3	77.0	80.2	86.3	93.2	51.3	52.7	57.3	71.9
4	76.3	77.3	80.2	86.3	93.2	51.7	52.7	57.5	71.8
5	76.6	77.3	80.3	86.3	93.2	51.7	52.9	57.4	71.9
6	76.6	77.5	80.3	86.4	93.2	51.9	52.9	57.5	71.9

Note that by the sixth iteration the change is less than 0.3°C; hence, we assume the temperature distribution is approximated by the last row of the table.

The heat rate per unit length can be determined by evaluating the heat rates in the x-direction for the control volumes about nodes 6, 7, and 8. From the schematic, find that

$$q' = q_1' + q_2' + q_3'$$

$$q' = k\Delta y \frac{T_8 - 25}{\Delta x} + k\Delta y \frac{T_7 - 25}{\Delta x} + k\frac{\Delta y}{2}\;\frac{T_6 - 25}{\Delta x}$$

Recognizing that $\Delta x = \Delta y$ and substituting numerical values, find

$$q' = 20\frac{\text{W}}{\text{m}\cdot\text{K}}\left[(57.5-25)+(52.9-25)+\frac{1}{2}(51.9-25)\right]\text{K}$$

$$q' = 1477\ \text{W/m}. \qquad <$$

COMMENTS: (1) Recognize that, while the temperature distribution may have been determined to a reasonable approximation, the uncertainty in the heat rate could be substantial. This follows since the heat rate is based upon a gradient and hence on temperature differences.

(2) Note that the initial guesses (k = 0) for the iteration are within 5°C of the final distribution. The geometry is simple enough that the guess can be very close. In some instances, a flux plot may be helpful and save labor in the calculation.

(3) In writing the FDEs, the iteration index (superscript k) was not included to simplify expression of the equations. However, the most recent value of $T_{m,n}$ is always used in the computations. Note that this system of FDEs is diagonally dominant and no rearrangement is required.