PROBLEM 2.40

KNOWN: Steady-state temperature distribution in a one-dimensional wall of thermal conductivity, $T(x) = Ax^3 + Bx^2 + Cx + D$.

FIND: Expressions for the heat generation rate in the wall and the heat fluxes at the two wall faces (x = 0,L).

ASSUMPTIONS: (1) Steady-state conditions, (2) One-dimensional heat flow, (3) Homogeneous medium.

ANALYSIS: The appropriate form of the heat diffusion equation for these conditions is

$$\frac{d^2T}{dx^2} + \frac{\dot{q}}{k} = 0 \quad \text{or} \quad \dot{q} = -k\frac{d^2T}{dx^2}.$$

Hence, the generation rate is

$$\dot{q} = -k\frac{d}{dx}\left[\frac{dT}{dx}\right] = -k\frac{d}{dx}\left[3Ax^2 + 2Bx + C + 0\right]$$

$$\dot{q} = -k\left[6Ax + 2B\right] \qquad <$$

which is linear with the coordinate x. The heat fluxes at the wall faces can be evaluated from Fourier's law,

$$q''_x = -k\frac{dT}{dx} = -k\left[3Ax^2 + 2Bx + C\right]$$

using the expression for the temperature gradient derived above. Hence, the heat fluxes are:

Surface x=0:

$$q''_x(0) = -kC \qquad <$$

Surface x=L:

$$q''_x(L) = -k\left[3AL^2 + 2BL + C\right]. \qquad <$$

COMMENTS: (1) From an overall energy balance on the wall, find

$$\dot{E}''_{in} - \dot{E}''_{out} + \dot{E}''_g = 0$$
$$q''_x(0) - q''_x(L) + \dot{E}''_g = (-kC) - (-k)\left[3AL^2 + 2BL + C\right] + \dot{E}''_g = 0$$
$$\dot{E}''_g = -3AkL^2 - 2BkL.$$

From integration of the volumetric heat rate, we can also find $\dot{E}''_g$ as

$$\dot{E}''_g = \int_0^L \dot{q}(x)dx = \int_0^L -k\left[6Ax + 2B\right]dx = -k\left[3Ax^2 + 2Bx\right]_0^L$$
$$\dot{E}''_g = -3AkL^2 - 2BkL.$$