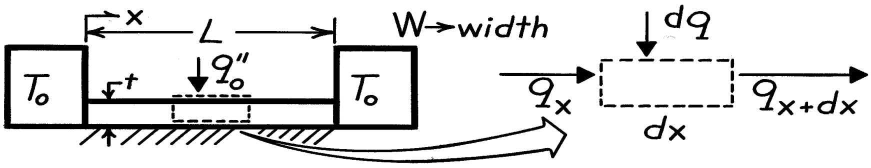
PROBLEM 3.112

KNOWN: Dimensions of a plate insulated on its bottom and thermally joined to heat sinks at its ends. Net heat flux at top surface.

FIND: (a) Differential equation which determines temperature distribution in plate, (b) Temperature distribution and heat loss to heat sinks.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state, (2) One-dimensional conduction in x (W,L>>t), (3) Constant properties, (4) Uniform surface heat flux, (5) Adiabatic bottom, (6) Negligible contact resistance.

ANALYSIS: (a) Applying conservation of energy to the differential control volume, $q_x + dq = q_{x+dx}$, where $q_{x+dx} = q_x + (dq_x/dx)\,dx$ and $dq = q''_o\,(W \cdot dx)$. Hence, $(dq_x/dx) - q''_o\,W = 0$. From Fourier's law, $q_x = -k(t \cdot W)\,dT/dx$. Hence, the differential equation for the temperature distribution is

$$-\frac{d}{dx}\left[ktW\,\frac{dT}{dx}\right] - q''_o\,W = 0 \qquad \frac{d^2T}{dx^2} + \frac{q''_o}{kt} = 0.$$ <

(b) Integrating twice, the general solution is,

$$T(x) = -\frac{q''_o}{2kt}x^2 + C_1\,x + C_2$$

and appropriate boundary conditions are $T(0) = T_o$, and $T(L) = T_o$. Hence, $T_o = C_2$, and

$$T_o = -\frac{q''_o}{2kt}L^2 + C_1 L + C_2 \qquad \text{and} \qquad C_1 = \frac{q''_o L}{2kt}.$$

Hence, the temperature distribution is

$$T(x) = -\frac{q''_o}{2kt}\left(x^2 - Lx\right) + T_o.$$ <

Applying Fourier's law at x = 0, and at x = L,

$$q(0) = -k(Wt)\,dT/dx)_{x=0} = -kWt\left[-\frac{q''_o}{kt}\right]\left[x - \frac{L}{2}\right]\Bigg|_{x=0} = -\frac{q''_o WL}{2}$$

$$q(L) = -k(Wt)dT/dx)_{x=L} = -kWt\left[-\frac{q''_o}{kt}\right]\left[x - \frac{L}{2}\right]\Bigg|_{x=L} = +\frac{q''_o WL}{2}$$

Hence the heat loss from the plates is $q = 2\left(q''_o WL/2\right) = q''_o WL$. <

COMMENTS: (1) Note signs associated with q(0) and q(L). (2) Note symmetry about x = L/2. Alternative boundary conditions are $T(0) = T_o$ and $dT/dx)_{x=L/2} = 0$.