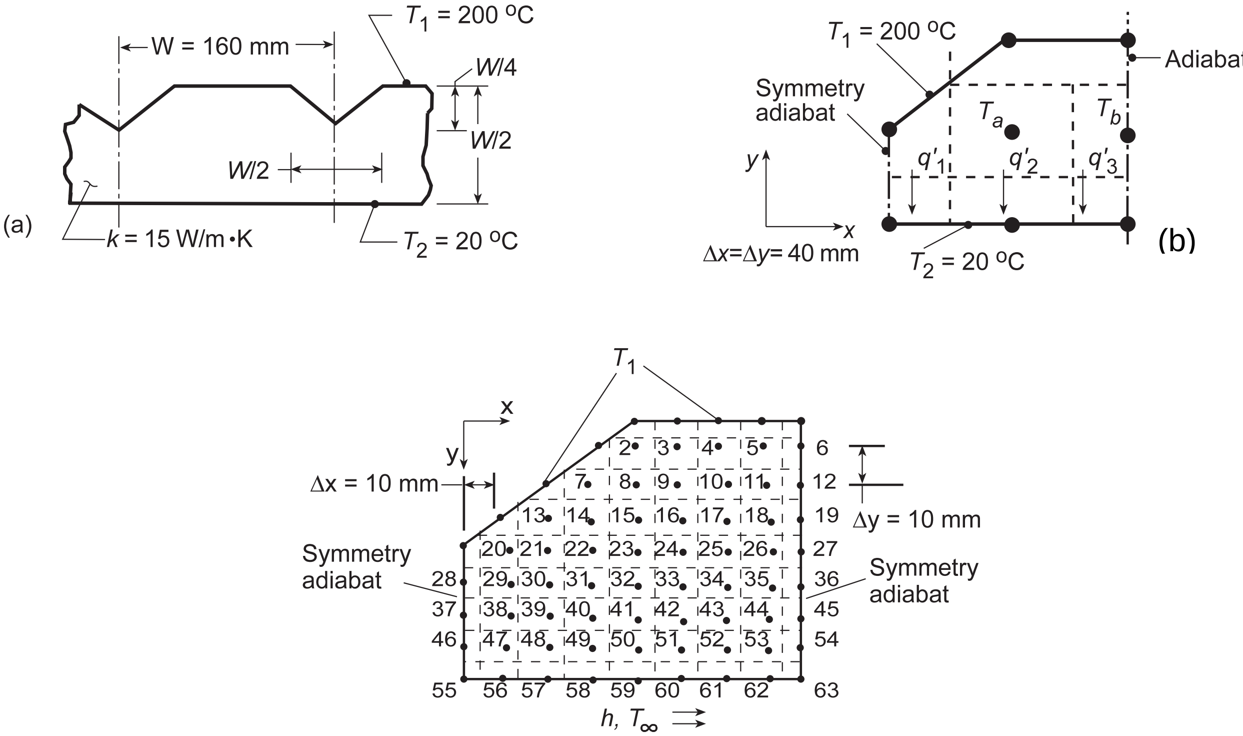
PROBLEM 4.74

KNOWN: Upper surface and grooves of a plate are maintained at a uniform temperature T_1, while the lower surface is maintained at T_2 or is exposed to a fluid at T_∞.

FIND: (a) Heat rate per width of groove spacing (w) for isothermal top and bottom surfaces using a finite-difference method with Δx = 40 mm, (b) Effect of grid spacing and convection at bottom surface.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state, two-dimensional conduction, (2) Constant properties.

ANALYSIS: (a) Using a space increment of Δx = 40 mm, the symmetrical section shown in schematic (b) corresponds to one-half the groove spacing. There exist only two interior nodes for which finite-difference equations must be written.

Node a:

$$4T_a - (T_1 + T_b + T_2 + T_1) = 0$$

$$4T_a - (200 + T_b + 20 + 200) = 0 \qquad \text{or} \qquad 4T_a - T_b = 420 \qquad (1)$$

Node b:

$$4T_b - (T_1 + T_a + T_2 + T_a) = 0$$

$$4T_b - (200 + 2T_a + 20) = 0 \qquad \text{or} \qquad -2T_a + 4T_b = 220 \qquad (2)$$

Multiply Eq. (2) by 2 and add to Eq. (1) to obtain

$7T_b = 860$ or $T_b = 122.9°C$

From Eq. (1),

$4T_a - 122.9 = 420$ or $T_a = (420 + 122.9)/4 = 135.7°C$.

The heat transfer through the symmetrical section is equal to the sum of heat flows through control volumes adjacent to the lower surface. From the schematic,

$$q' = q_1' + q_2' + q_3' = k\left(\frac{\Delta x}{2}\right)\frac{T_1 - T_2}{\Delta y} + k(\Delta x)\frac{T_a - T_2}{\Delta y} + k\left(\frac{\Delta x}{2}\right)\frac{T_b - T_2}{\Delta y}.$$

Continued...

Noting that $\Delta x = \Delta y$, regrouping and substituting numerical values, find

$$q' = k\left[\frac{1}{2}(T_1 - T_2) + (T_a - T_2) + \frac{1}{2}(T_b - T_2)\right]$$

$$q' = 15\,\text{W/m}\cdot\text{K}\left[\frac{1}{2}(200-20) + (135.7-20) + \frac{1}{2}(122.9-20)\right] = 3.86\,\text{kW/m}\,.$$

For the full groove spacing, q'_{total} = 2 × 3.86 kW/m = 7.72 kW/m. $<$

(b) Using the *Finite-Difference Equations* option from the *Tools* portion of the IHT menu, the following two-dimensional temperature field was computed for the grid shown in schematic (b), where x and y are in mm and the nodal temperatures are in °C. Nodes 2-54 are interior nodes, with those along the symmetry adiabats characterized by $T_{m-1,n} = T_{m+1,n}$, while nodes 55-63 lie on a plane surface.

y\x	0	10	20	30	40	50	60	70	80
0					200	200	200	200	200
10				200	191	186.6	184.3	183.1	182.8
20			200	186.7	177.2	171.2	167.5	165.5	164.8
30		200	182.4	169.5	160.1	153.4	149.0	146.4	145.5
40	200	175.4	160.3	148.9	140.1	133.5	128.7	125.7	124.4
50	141.4	134.3	125.7	118.0	111.6	106.7	103.1	100.9	100.1
60	97.09	94.62	90.27	85.73	81.73	78.51	76.17	74.73	74.24
70	57.69	56.83	55.01	52.95	51.04	49.46	48.31	47.60	47.36
80	20	20	20	20	20	20	20	20	20

The foregoing results were computed for h = 10^7 W/m^2·K ($h \to \infty$) and T_∞ = 20°C, which is tantamount to prescribing an isothermal bottom surface at 20°C. Agreement between corresponding results for the coarse and fine grids is surprisingly good (T_a = 135.7°C $\leftrightarrow$ T_{23} = 140.1°C; T_b = 122.9°C $\leftrightarrow$ T_{27} = 124.4°C). The heat rate is

$$q' = 2\times k\left[(T_{46}-T_{55})/2 + (T_{47}-T_{56}) + (T_{48}-T_{57}) + (T_{49}-T_{58}) + (T_{50}-T_{59}) + (T_{51}-T_{60}) + (T_{52}-T_{61}) + (T_{53}-T_{62}) + (T_{54}-T_{63})/2\right]$$

$$q' = 2\times 15\,\text{W/m}\cdot\text{K}\left[18.84+36.82+35.00+32.95+31.04+29.46+28.31+27.6+13.68\right]^\circ\text{C} = 7.61\,\text{kW/m}$$ $<$

The agreement with q' = 7.72 kW/m from the coarse grid of part (a) is excellent and a fortuitous consequence of compensating errors. With reductions in the convection coefficient from $h \to \infty$ to h = 1000, 200 and 5 W/m^2·K, the corresponding increase in the thermal resistance reduces the heat rate to values of 6.03, 3.28 and 0.14 kW/m, respectively. With decreasing h, there is an overall increase in nodal temperatures, as, for example, from 191°C to 199.8°C for T_2 and from 20°C to 196.9°C for T_{55}.

NOTE TO INSTRUCTOR: To reduce computational effort, while achieving the same educational objectives, the problem statement has been changed to allow for convection at the bottom, rather than the top, surface.