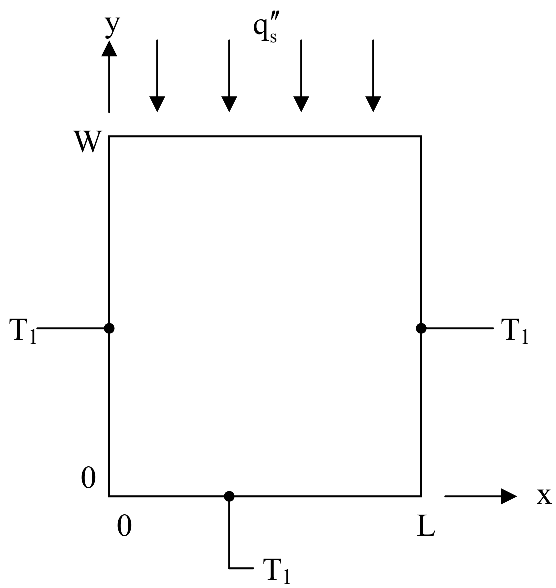
PROBLEM 4.5

KNOWN: Boundary conditions on four sides of a rectangular plate.

FIND: Temperature distribution.

SCHEMATIC:



ASSUMPTIONS: (1) Two-dimensional, steady-state conduction, (2) Constant properties.

ANALYSIS: This problem differs from the one solved in Section 4.2 only in the boundary condition at the top surface. Defining $\theta = T - T_\infty$, the differential equation and boundary conditions are

$$\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} = 0$$

$$\theta(0, y) = 0 \qquad \theta(L, y) = 0 \qquad \theta(x, 0) = 0 \qquad k\left.\frac{\partial\theta}{\partial y}\right|_{y=W} = q''_s \qquad \text{(1a,b,c,d)}$$

The solution is identical to that in Section 4.2 through Equation (4.11),

$$\theta = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{L} \sinh\frac{n\pi y}{L} \qquad (2)$$

To determine C_n, we now apply the top surface boundary condition, Equation (1d). Differentiating Equation (2) yields

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PROBLEM 4.5 (Cont.)

$$\left.\frac{\partial\theta}{\partial y}\right|_{y=W} = \sum_{n=1}^{\infty} C_n \frac{n\pi}{L}\sin\frac{n\pi x}{L}\cosh\frac{n\pi W}{L} \tag{3}$$

Substituting this into Equation (1d) results in

$$\frac{q_s''}{k} = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L} \tag{4}$$

where $A_n = C_n(n\pi/L)\cosh(n\pi W/L)$. The principles expressed in Equations (4.13) through (4.16) still apply, but now with reference to Equation (4) and Equation (4.14), we should choose $f(x) = q_s''/k$, $g_n(x) = \sin\frac{n\pi x}{L}$. Equation (4.16) then becomes

$$A_n = \frac{\frac{q_s''}{k}\int_0^L \sin\frac{n\pi x}{L}dx}{\int_0^L \sin^2\frac{n\pi x}{L}dx} = \frac{q_s''}{k}\frac{2}{\pi}\frac{(-1)^{n+1}+1}{n}$$

Thus

$$C_n = 2\frac{q_s''L}{k}\frac{(-1)^{n+1}+1}{n^2\pi^2\cosh(n\pi W/L)} \tag{5}$$

The solution is given by Equation (2) with C_n defined by Equation (5).