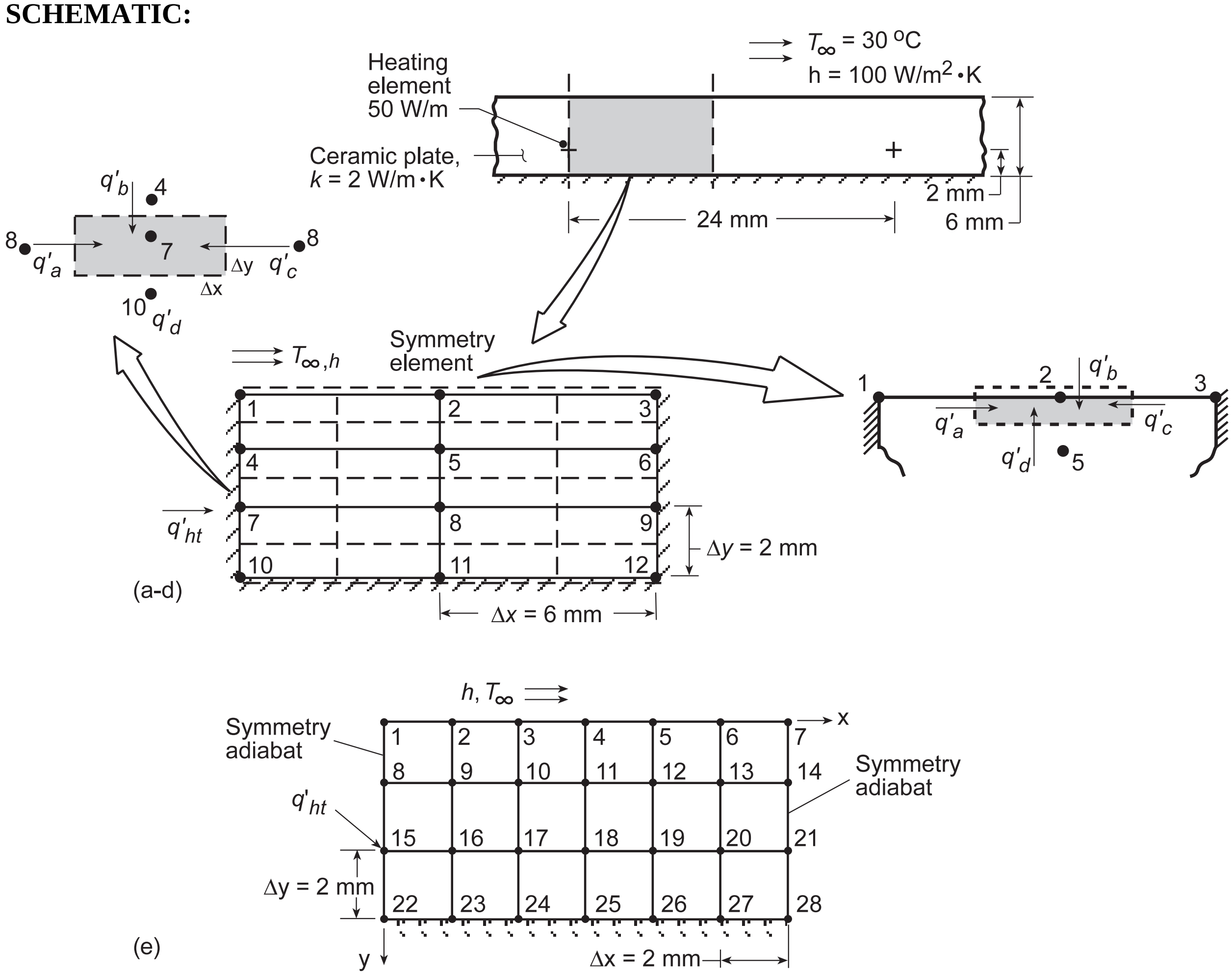
PROBLEM 4.86

KNOWN: Electrical heating elements with known dissipation rate embedded in a ceramic plate of known thermal conductivity; lower surface is insulated, while upper surface is exposed to a convection process.

FIND: (a) Temperature distribution within the plate using prescribed grid spacing, (b) Sketch isotherms to illustrate temperature distribution, (c) Heat loss by convection from exposed surface (compare with element dissipation rate), (d) Advantage, if any, in not setting $\Delta x = \Delta y$, (e) Effect of grid size and convection coefficient on the temperature field.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state, two-dimensional conduction in ceramic plate, (2) Constant properties, (3) No internal generation, except for Node 7 (or Node 15 for part (e)), (4) Heating element approximates a line source of negligible wire diameter.

ANALYSIS: (a) The prescribed grid for the symmetry element shown above consists of 12 nodal points. Nodes 1-3 are points on a surface experiencing convection; nodes 4-6 and 8-12 are interior nodes. Node 7 is a special case of the interior node having a generation term; because of symmetry, $q'_{ht} = 25$ W/m. The finite-difference equations are derived as follows:

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PROBLEM 4.86 (Cont.)

Surface Node 2. From an energy balance on the prescribed control volume with $\Delta x/\Delta y = 3$,

$\dot{E}_{in} - \dot{E}_{out} = q'_a + q'_b + q'_c + q'_d = 0$;

$$k\frac{\Delta y}{2}\frac{T_1 - T_2}{\Delta x} + h\Delta x\left(T_\infty - T_2\right) + k\frac{\Delta y}{2}\frac{T_3 - T_2}{\Delta x} + k\Delta x\frac{T_5 - T_2}{\Delta y} = 0 .$$

Regrouping, find

$$T_2\left(1 + 2N\frac{\Delta x}{\Delta y} + 1 + 2\left(\frac{\Delta x}{\Delta y}\right)^2\right) = T_1 + T_3 + 2\left(\frac{\Delta x}{\Delta y}\right)^2 T_5 + 2N\frac{\Delta x}{\Delta y}T_\infty$$

where $N = h\Delta x/k = 100\ W/m^2 \cdot K \times 0.006\ m/2\ W/m \cdot K = 0.30\ K$. Hence, with $T_\infty = 30°C$,

$$T_2 = 0.04587T_1 + 0.04587T_3 + 0.82569T_5 + 2.4771 \qquad (1)$$

From this FDE, the forms for nodes 1 and 3 can also be deduced.

Interior Node 7. From an energy balance on the prescribed control volume, with $\Delta x/\Delta y = 3$, $\dot{E}'_{in} - \dot{E}'_g = 0$, where $\dot{E}'_g = 2\,q'_{ht}$ and E'_{in} represents the conduction terms. Hence,

$q'_a + q'_b + q'_c + q'_d + 2q'_{ht} = 0$, or

$$k\Delta y\frac{T_8 - T_7}{\Delta x} + k\Delta x\frac{T_4 - T_7}{\Delta y} + k\Delta y\frac{T_8 - T_7}{\Delta x} + k\Delta x\frac{T_{10} - T_7}{\Delta y} + 2q'_{ht} = 0$$

Regrouping,

$$T_7\left[1 + \left(\frac{\Delta x}{\Delta y}\right)^2 + 1 + \left(\frac{\Delta x}{\Delta y}\right)^2\right] = T_8 + \left(\frac{\Delta x}{\Delta y}\right)^2 T_4 + T_8 + \left(\frac{\Delta x}{\Delta y}\right)^2 T_{10} + \frac{2q'_{ht}}{k}\left(\frac{\Delta x}{\Delta y}\right)$$

Recognizing that $\Delta x/\Delta y = 3$, $q'_{ht} = 25$ W/m and $k = 2$ W/m·K, the FDE is

$$T_7 = 0.0500T_8 + 0.4500T_4 + 0.0500T_8 + 0.4500T_{10} + 3.7500 \qquad (2)$$

The FDEs for the remaining nodes may be deduced from this form. Following the procedure described in Appendix D for the Gauss-Seidel method, the system of FDEs has the form:

$$T_1^k = 0.09174T_2^{k-1} + 0.8257T_4^{k-1} + 2.4771$$

$$T_2^k = 0.04587T_1^k + 0.04587T_3^{k-1} + 0.8257T_5^{k-1} + 2.4771$$

$$T_3^k = 0.09174T_2^k + 0.8257T_6^{k-1} + 2.4771$$

$$T_4^k = 0.4500T_1^k + 0.1000T_5^{k-1} + 0.4500T_7^{k-1}$$

$$T_5^k = 0.4500T_2^k + 0.0500T_4^k + 0.0500T_6^{k-1} + 0.4500T_8^{k-1}$$

$$T_6^k = 0.4500T_3^k + 0.1000T_5^k + 0.4500T_9^{k-1}$$

$$T_7^k = 0.4500T_4^k + 0.1000T_8^{k-1} + 0.4500T_{10}^{k-1} + 3.7500$$

$$T_8^k = 0.4500T_5^k + 0.0500T_7^k + 0.0500T_9^{k-1} + 0.4500T_{11}^{k-1}$$

$$T_9^k = 0.4500T_6^k + 0.1000T_8^k + 0.4500T_{12}^{k-1}$$

$$T_{10}^k = 0.9000T_7^k + 0.1000T_{11}^{k-1}$$

$$T_{11}^k = 0.9000T_8^k + 0.0500T_{10}^{k-1} + 0.0500T_{12}^{k-1}$$

$$T_{12}^k = 0.9000T_9^k + 0.1000T_{11}^k$$

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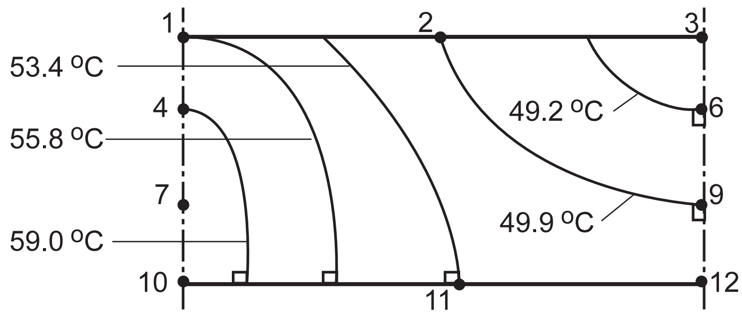
Note the use of the superscript k to denote the level of iteration. Begin the iteration procedure with rational estimates for T_i (k = 0) and prescribe the convergence criterion as $\varepsilon \leq 0.1$ K.

k/T_i	1	2	3	4	5	6	7	8	9	10	11	12
0	55.0	50.0	45.0	61.0	54.0	47.0	65.0	56.0	49.0	60.0	55.0	50.0
1	57.4	51.7	46.0	60.4	53.8	48.1	63.5	54.6	49.6	62.7	54.8	50.1
2	57.1	51.6	46.9	59.7	53.2	48.7	64.3	54.3	49.9	63.4	54.5	50.4
∞	55.80	49.93	47.67	59.03	51.72	49.19	63.89	52.98	50.14	62.84	53.35	50.46

The last row with k = ∞ corresponds to the solution obtained by matrix inversion. It appears that at least 20 iterations would be required to satisfy the convergence criterion using the Gauss-Seidel method.

(b) Selected isotherms are shown in the sketch of the nodal network.



Note that the isotherms are normal to the adiabatic surfaces.

(c) The heat loss by convection can be expressed as

$$q'_{conv} = h\left[\frac{1}{2}\Delta x\left(T_1 - T_\infty\right) + \Delta x\left(T_2 - T_\infty\right) + \frac{1}{2}\Delta x\left(T_3 - T_\infty\right)\right]$$

$$q'_{conv} = 100\,\mathrm{W/m^2 \cdot K} \times 0.006\,\mathrm{m}\left[\frac{1}{2}\left(55.80 - 30\right) + \left(49.93 - 30\right) + \frac{1}{2}\left(47.67 - 30\right)\right] = 25.00\,\mathrm{W/m}. \quad <$$

As expected, the heat loss by convection is equal to the heater element dissipation. This follows from the conservation of energy requirement.

(d) For this situation, choosing $\Delta x = 3\Delta y$ was advantageous from the standpoint of precision and effort. If we had chosen $\Delta x = \Delta y = 2$ mm, there would have been 28 nodes, doubling the amount of work, but with improved precision.

(e) Examining the effect of grid size by using the *Finite-Difference Equations* option from the *Tools* portion of the IHT Menu, the following temperature field was computed for $\Delta x = \Delta y = 2$ mm, where x and y are in mm and the temperatures are in °C.

y\x	0	2	4	6	8	10	12
0	55.04	53.88	52.03	50.32	49.02	48.24	47.97
2	58.71	56.61	54.17	52.14	50.67	49.80	49.51
4	66.56	59.70	55.90	53.39	51.73	50.77	50.46
6	63.14	59.71	56.33	53.80	52.09	51.11	50.78

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PROBLEM 4.86 (Cont.)

Agreement with the results of part (a) is excellent, except in proximity to the heating element, where T_{15} = 66.6°C for the fine grid exceeds T_7 = 63.9°C for the coarse grid by 2.7°C.

For h = 10 W/m^2·K, the maximum temperature in the ceramic corresponds to T_{15} = 254°C, and the heater could still be operated at the prescribed power. With h = 10 W/m^2·K, the critical temperature of T_{15} = 400°C would be reached with a heater power of approximately 82 W/m.

COMMENTS: (1) The method used to obtain the rational estimates for T_i (k = 0) in part (a) is as follows. Assume 25 W/m is transferred by convection uniformly over the surface; find $\overline{T}_{surf} \approx 50°C$. Set T_2 = 50°C and recognize that T_1 and T_3 will be higher and lower, respectively. Assume 25 W/m is conducted uniformly to the outer nodes; find $T_5 - T_2 \approx 4°C$. For the remaining nodes, use intuition to guess reasonable values. (2) In selecting grid size (and whether $\Delta x = \Delta y$), one should consider the region of largest temperature gradients. Predicted values of the maximum temperature in the ceramic will be very sensitive to the grid resolution.

NOTE TO INSTRUCTOR: Although the problem statement calls for calculations with $\Delta x = \Delta y$ = 1 mm, the instructional value and benefit-to-effort ratio are small. Hence, consideration of this grid size is not recommended.