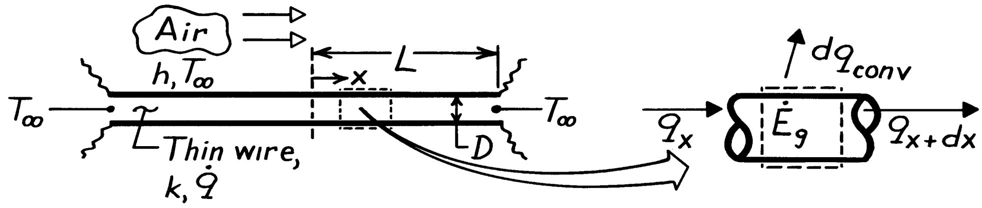
PROBLEM 3.116

KNOWN: Thermal conductivity, diameter and length of a wire which is annealed by passing an electrical current through the wire.

FIND: (a) Steady-state temperature distribution along wire, (b) Maximum wire temperature, (c) Average wire temperature.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) One-dimensional conduction along the wire, (3) Constant properties, (4) Negligible radiation, (5) Uniform convection coefficient h.

ANALYSIS: (a) Applying conservation of energy to a differential control volume,

$$q_x + \dot{E}_g - dq_{conv} - q_{x+dx} = 0$$

$$q_{x+dx} = q_x + \frac{dq_x}{dx}dx \qquad q_x = -k\left(\pi\ D^2/4\right)dT/dx$$

$$dq_{conv} = h\left(\pi\ D\ dx\right)\ \left(T - T_\infty\right) \qquad \dot{E}_g = \dot{q}\left(\pi\ D^2/4\right)dx.$$

Hence,

$$k\left(\pi\ D^2/4\right)\ \frac{d^2T}{dx^2}dx + \dot{q}\left(\pi\ D^2/4\right)dx - h\left(\pi\ Ddx\right)\ \left(T - T_\infty\right) = 0$$

or, with $\theta \equiv T - T_\infty$, $\qquad \dfrac{d^2\theta}{dx^2} - \dfrac{4h}{kD}\theta + \dfrac{\dot{q}}{k} = 0$

The solution (general and particular) to this nonhomogeneous equation is of the form

$$\theta = C_1\ e^{mx} + C_2\ e^{-mx} + \frac{\dot{q}}{km^2}$$

where $m^2 = (4h/kD)$. The boundary conditions are:

$$\left.\frac{d\theta}{dx}\right|_{x=0} = 0 = m\ C_1\ e^0 - mC_2\ e^0 \rightarrow \qquad C_1 = C_2$$

$$\theta\left(L\right) = 0 = C_1\left(e^{mL} + e^{-mL}\right) + \frac{\dot{q}}{km^2} \rightarrow \qquad C_1 = \frac{-\dot{q}/km^2}{e^{mL} + e^{-mL}} = C_2$$

Continued...

The temperature distribution has the form

$$T = T_\infty - \frac{\dot{q}}{km^2}\left[\frac{e^{mx}+e^{-mx}}{e^{mL}+e^{-mL}} - 1\right] = T_\infty - \frac{\dot{q}}{km^2}\left[\frac{\cosh mx}{\cosh mL} - 1\right]. \qquad <$$

(b) The maximum wire temperature exists at $x = 0$. Hence,

$$T_{\max} = T(x=0) = T_\infty - \frac{\dot{q}}{km^2}\left[\frac{\cosh(0)}{\cosh(mL)} - 1\right] = T_\infty - \frac{\dot{q}}{km^2}\left[\frac{1}{\cosh(mL)} - 1\right] \qquad <$$

(c) The average wire temperature may be obtained by evaluating the expression

$$\overline{T} = \frac{1}{L}\int_{x=0}^{L} T(x)dx = \frac{1}{L}\int_{x=0}^{L}\left[T_\infty - \frac{\dot{q}}{km^2}\left[\frac{\cosh(mx)}{\cosh(mL)} - 1\right]\right]dx$$

$$= T_\infty + \frac{\dot{q}}{km^2} - \tanh(mL)\frac{\dot{q}}{Lkm^3} \qquad <$$

COMMENTS: (1) This process is commonly used to anneal wire and spring products. It is also used for flow measurement based upon the principle that the maximum or average wire temperature varies with the value of *m* and, hence, the convective heat transfer coefficient *h* and, ultimately, the fluid velocity. (2) To check the result of part (a), note that $T(L) = T(-L) = T_\infty$.