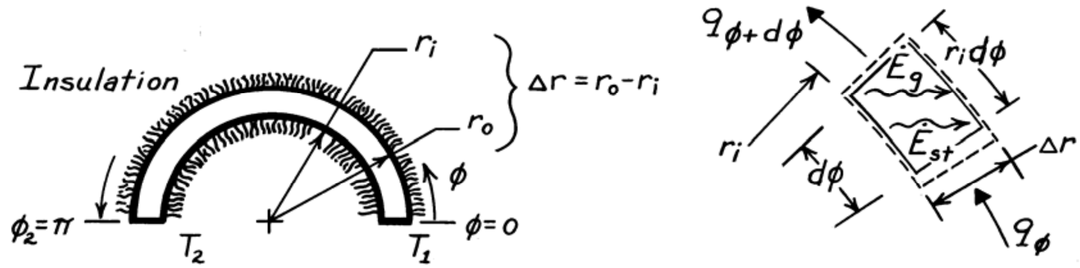
PROBLEM 2.43

KNOWN: Cylindrical system with negligible temperature variation in the r,z directions.

FIND: (a) Heat equation beginning with a properly defined control volume, (b) Temperature distribution $T(\phi)$ for steady-state conditions with no internal heat generation and constant properties, (c) Heat rate for Part (b) conditions.

SCHEMATIC:



ASSUMPTIONS: (1) T is independent of r,z, (2) $\Delta r = (r_o - r_i) << r_i$.

ANALYSIS: (a) Define the control volume as $V = r_i d\phi \cdot \Delta r \cdot L$ where L is length normal to page. Apply the conservation of energy requirement, Eq. 1.12c,

$$\dot{E}_{in} - \dot{E}_{out} + \dot{E}_g = \dot{E}_{st} \qquad q_\phi - q_{\phi+d\phi} + \dot{q}V = \rho V c \frac{\partial T}{\partial t} \tag{1,2}$$

where $$q_\phi = -k(\Delta r \cdot L)\frac{\partial T}{r_i \partial \phi} \qquad q_{\phi+d\phi} = q_\phi + \frac{\partial}{\partial \phi}(q_\phi)d\phi. \tag{3,4}$$

Eqs. (3) and (4) follow from Fourier's law, Eq. 2.1, and from Eq. 2.25, respectively. Combining Eqs. (3) and (4) with Eq. (2) and canceling like terms, find

$$\frac{1}{r_i^2}\frac{\partial}{\partial \phi}\left(k\frac{\partial T}{\partial \phi}\right) + \dot{q} = \rho c \frac{\partial T}{\partial t}. \tag{5}$$ <

Since temperature is independent of r and z, this form agrees with Eq. 2.26.

(b) For steady-state conditions with $\dot{q} = 0$, the heat equation, (5), becomes

$$\frac{d}{d\phi}\left[k\frac{dT}{d\phi}\right] = 0. \tag{6}$$

With constant properties, it follows that $dT/d\phi$ is constant which implies $T(\phi)$ is linear in ϕ. That is,

$$\frac{dT}{d\phi} = \frac{T_2 - T_1}{\phi_2 - \phi_1} = +\frac{1}{\pi}(T_2 - T_1) \quad \text{or} \quad T(\phi) = T_1 + \frac{1}{\pi}(T_2 - T_1)\phi. \tag{7,8}$$ <

(c) The heat rate for the conditions of Part (b) follows from Fourier's law, Eq. (3), using the temperature gradient of Eq. (7). That is,

$$q_\phi = -k(\Delta r \cdot L)\frac{1}{r_i}\left[+\frac{1}{\pi}(T_2 - T_1)\right] = -k\left[\frac{r_o - r_i}{\pi r_i}\right]L(T_2 - T_1). \tag{9}$$ <

COMMENTS: Note the expression for the temperature gradient in Fourier's law, Eq. (3), is $\partial T/r_i\partial\phi$ not $\partial T/\partial\phi$. For the conditions of Parts (b) and (c), note that q_ϕ is independent of ϕ; this is first indicated by Eq. (6) and confirmed by Eq. (9).